Uniform Convergence

We first see that integration and uniform convergence play well together.

Theorem 1. *Let f be a Riemann integrable function on $[a,b]$. Let $f_1, f_2, \ldots$ be a sequence of Riemann integrable on $[a,b]$ functions with $\lim_{n\to\infty} f_n = f$ uniformly. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

Problem 1. Prove this. □

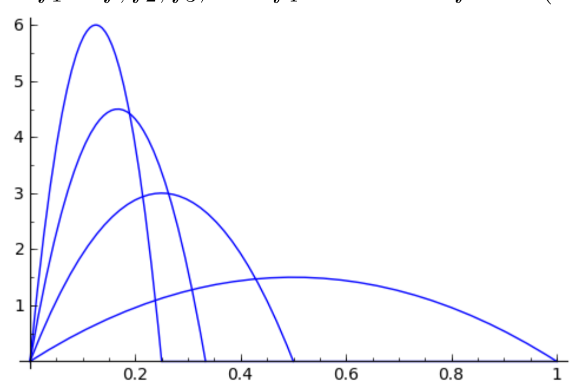
Pointwise convergence and integration do not play well together. To give an example let $f(x)$ be continuous and defined on $[0,1]$ with

$$f(x) \ge 0, \qquad f(0) = f(1), \qquad \text{and} \qquad \int_0^1 f(x)\,dx = 1$$

and for positive integers n on $[0,1]$ by

$$f_n(x) = \begin{cases} nf(nx), & 0 \le x < 1/n; \\ 0, & 1/n \le x \le 1. \end{cases}$$

The figure shows $f_1 = f$, f_2, f_3, and f_4 in the case $f = 6x(1-x)$.



Problem 2. With this set up

(a) Show that $\lim_{n\to\infty} f_n(x) = 0$ pointwise on $[0,1]$.
(b) Compute $\int_0^1 f_n(x)\,dx$.
(c) Compute $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx$.
(d) Explain why this shows there is no version of Theorem 1 with uniform convergence replaced by pointwise convergence. □

The interaction between uniform convergence and differentiation is more complicated.

Theorem 2. *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuously differentiable functions on an open interval I such that for some continuously differentiable* *function f on I we have*

$$\lim_{n\to\infty} f_n'(x) = f'(x)$$

and for some point $x_0 \in I$

$$\lim_{n\to\infty} f_n(x_0) = f(x_0).$$

Then

$$f(x) = \lim_{n\to\infty} f_n(x)$$

for all $x \in I$.

Problem 3. Prove this. *Hint:* Theorem 1 and that

$$f_n(x) = f_n(x_0) + \int_{x_0}^{x} f_n'(x)\,dx. \qquad \square$$

We now translate these results into theorems about infinite series of functions.

Theorem 3. *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuous functions on $[a, b]$ such that*

$$f(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges uniformly on $[a, b]$. Then f is continuous on $[a, b]$ and

$$\int_a^b f(x)\,dx = \sum_{k=1}^{\infty} \int_a^b f_k(x)\,dx.$$

Problem 4. Prove this. $\square$

Theorem 4. *Let $f_1, f_2, f_3, \ldots$ be a sequence of continuously differentiable functions on the open interval I such that the series of derivatives*

$$\sum_{k=1}^{\infty} f_k'$$

converges uniformly on I. Assume there is a point $x_0 \in I$ such that

$$\sum_{k=1}^{\infty} f_k(x_0)$$

converges. Then

$$S(x) = \sum_{k=1}^{\infty} f_k(x)$$

converges pointwise to a continuously differentiable function S on I and

$$S'(x) = \sum_{k=1}^{\infty} f_k'(x)$$

Problem 5. Prove this. *Hint:* Let $S_n = \sum_{k=1}^n f_n$ and

$$g(x) = \sum_{k=1}^{\infty} f_k'(x).$$

As the series for g is a uniformly convergent series of continuous function g is continuous.

$$\lim_{n\to\infty} S_n' = \lim_{n\to\infty} \sum_{k=1}^{n} f_n' = g$$

and this limit is uniform on $[a, b]$. Therefore, by Theorem 2 (or rather its proof) we have that the limit

$$S(x) := \lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \left(S_n(x_0) + \int_{x_0}^{x} S_n'(t)\,dt \right) = \sum_{k=1}^{\infty} f_k(x_0) + \int_{x_0}^{x} g(t)\,dt$$

exists and

$$\lim_{n\to\infty} S_n(x) = \frac{d}{dx} \left(\sum_{k=1}^{\infty} f_k(x_0) + \int_{x_0}^{x} g(t)\,dt \right) = g(x).$$

The rest should be easy. □

Proposition 5. *Let the power series*

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

have radius of convergence $r > 0$. Then for any r_0 with $0 < r_0 < r$ the series

$$f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k, \qquad \textit{and} \qquad f^*(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1}$$

converge uniformly on $(-r_0, r_0)$.

Problem 6. Prove this. *Hint:* Look back at the notes on series. The Weierstrass M test will be useful. □

Problem 7. Use Proposition 5 and Theorem 4 to give another prove of the theorem on the term wise differentiation of power series. That is if $f(x) = \sum_{k=0}^{\infty} a_k (x - x_0)^k$ has radius of convergence r, then $f(x)$ is differentiable on $(x_0 - r, x_0 + r)$ and on this interval the derivative is commuted by the termwise differentiation of the series for f, that is $f'(x) = \sum_{k=0}^{\infty} k a_k (x - x_0)^{k-1}$.