Mathematics 555 Test #1 Name: Answer Key

1. Let f be a function defined on an open interval and $x_0 \in I$.

(a) State what it means the derivative $f'(x_0)$ to exist.

Solution: This means that the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$$

exits. □

(b) Prove that if f is differentiable at x_0, then f is continuous at x_0.

Solution: To show that f is continuous at x_0 it is enough show

$$\lim_{x\to x_0} f(x) = f(x_0).$$

We do this by our standard trick of finding an artfully complication of the function $f(x)$.

$$\begin{aligned}
\lim_{x\to x_0} f(x) &= \lim_{x\to x_0}\left(f(x_0) + \frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right) \\
&= f(x_0) + \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}(x-x_0)\right) && \text{(As } f(x_0) \text{ is constant.)} \\
&= f(x_0) + \left(\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\right)\left(\lim_{x\to x_0}(x-x_0)\right) \\
&= f(x_0) + f'(x_0)(0) \\
&= f(x_0)
\end{aligned}$$

where we have used a theorem about the product of limits that exist. □

(c) Prove the product rule: If f and g are both differentiable at x_0 then so is the product $p(x) = f(x)g(x)$ and $p'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

Solution: We are given that the limits

$$f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \qquad \text{and} \qquad g'(x_0) = \lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0}$$

exist. Thus (and again we use an artfully complication)

$$\begin{aligned}
p'(x_0) &= \lim_{x\to x_0}\frac{p(x)-p(x_0)}{x-x_0} \\
&= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0} \\
&= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}g(x) + f(x_0)\frac{g(x)-g(x_0)}{x-x_0}\right) \\
&= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}\lim_{x\to x_0} g(x) + f(x_0)\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} \\
&= f'(x_0)g(x_0) + f(x_0)g'(x_0).
\end{aligned}$$

We have used that $\lim_{x\to x_0} g(x) = g(x_0)$ as g is differentiable at x_0 and thus continuous at x_0. □

2. (a) State Rôlle's theorem.

Solution: If f continuous on the closed interval $[a, b]$, differentable on the open interval (a, b), and $f(a) = f(b)$, then there is a $\xi \in (a, b)$ with such that $f'(\xi) = 0$. □

(b) Prove that if h is twice differentiable on an interval (a, b) and there are points $x_1, x_2, x_3 \in (a, b)$ with $x_1 < x_2 < x_3$ and $h(x_1) = h(x_2) = h(x_3) = 0$, then there is a point $\xi \in (x_1, x_3)$ with $h''(\xi) = 0$.

Solution: By Rôlle's theorem there is ξ_1 between x_1 and x_2 with $h'(\xi_1) = 0$ and a ξ_2 between x_2 and x_3 with $h'(\xi_2) = 0$. The function h' is differentable on (ξ_1, ξ_2) so by another application of Rôlle's theorem there is a $\xi \in (\xi_1, \xi_2) \subseteq (x_1, x_3)$ with $h''(\xi) = (h')'(\xi) = 0$. □

(c) Prove if f and g are twice differentiable on the open interval (a, b) and there are $x_1, x_2, x_3 \in (a, b)$ with $x_1 < x_2 < x_3$ and

$$f(x_1) = g(x_1), \qquad f(x_2) = g(x_2), \qquad f(x_3) = g(x_3)$$

then there is a point $\xi \in (x_1, x_2)$ with $f''(\xi) = g''(\xi) = 0$.

Solution: This follows more or less directly form part (b). Let $h = f - g$. Then $h(x_1) = h(x_2) = h(x_3) = 0$. Thus by (b) there is a $\xi \in (x_1, x_3)$ with $h''(\xi) = (f - g)''(\xi) = f''(\xi) - g''(\xi) = 0$. Thus $f''(\xi) = g''(\xi)$. □

3. (a) State the mean value theorem.

Solution: If f continuous on the closed interval $[a, b]$ and differentable on the open interval (a, b), then there is a $\xi \in (a, b)$ with such that

$$f(b) - f(a) = f'(\xi)(b - a).$$

□

(b) Let f be a function defined on $\mathbb{R}$ such that for all x the inequality

$$|f'(x)| \le 42$$

hold for all x. Show that for all $x, y \in \mathbb{R}$

$$|f(x) - f(y)| \le 42|x - y|.$$

Solution: By the mean value theorem there is a ξ between x and y such that $f(x) - f(y) = f'(\xi)(x - y)$. Using this and $|f'(\xi)| \le 42$ gives

$$|f(x) - f(y)| = |f'(\xi)(x - y)| = |f'(\xi)||x - y| \le 42|x - y|$$

as required. □

4. (a) State Taylor's theorem with Lagrange's form of the remainder.

Solution: If f is $n + 1$ times differentable on an open interval I and $a, x \in I$ then there is a ξ between a and x such that

$$f(x) = f(a) + f'(a)(x - a) + \frac{f'(a)}{2!}(x - a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x - a)^n + \frac{f^{(n+1)}(\xi)}{(n + 1)!}(x - a)^{n+1}.$$

□

(b) What are the first three terms of the Taylor expansion of $f(x) = \sqrt{1+x}$ about $x = 0$.

Solution: We have

$$\begin{aligned} f(x) &= (1+x)^{1/2} & f(0) &= 1 \\ f'(x) &= \frac{1}{2}(1+x)^{-1/2} & f'(0) &= \frac{1}{2} \\ f''(x) &= \frac{-1}{4}(1+x)^{-3/2} & f''(0) &= \frac{-1}{4} \end{aligned}$$

and therefore the first three terms of the Taylor expansion are

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \cdots = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots$$

□

(c) What is the Taylor series for $\sin(x)$ about $x = 0$. (You do not have to derive it, you just have to state it.)

$$\sin(x) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

□

5. (a) State the fundamental theorem of calculus.

Solution: If f is Riemann integrable on $[a, b]$, F is defined by

$$F(x) = \int_a^x f(t)\, dt,$$

and f is continuous at x_0, then F is differentiable at x_0 and

$$F'(x_0) = f(x_0).$$

□

(b) Prove that if f is continuous on $[a, b]$ there is a $\xi \in (a, b)$ such that

$$f(\xi) = \frac{1}{b-a}\int_a^b f(t)\, dt$$

Hint: One way is to apply the mean value theorem to the function

$$F(x) = \int_a^x f(t)\, dt.$$

Solution: As f is continuous on $[a, b]$ the function F is differentiable at all points of (a, b) by the fundamental theorem of calculus. Thus the mean value theorem applies and we have that there is a $\xi \in (a, b)$ with

(1) $$F(b) - F(a) = F'(\xi)(b - a).$$

But

$$F(b) - F(a) = \int_a^b f(t)\, dt - \int_a^a f(t)\, dt = \int_a^b f(t)\, dt - 0 = \int_a^b f(t)\, dt.$$

And by the fundamental theorem of calculus

$$F'(\xi) = f(\xi).$$

Using these facts in (1) gives

$$\int_a^b f(t)\,dt = f(\xi)(b-a).$$

Dividing by $(b-a)$ now gives the result. $\square$