Mathematics 555 Test #2 Name:______________________

Show your work! Answers that do not have a justification will receive no credit.

1. Fund the radius of convergence of the following two power series.

(a) $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k x^{3k+1}}{2^k(k+1)}$.

Solution: We use the ratio test:

$$\text{ratio} = \lim_{k\to\infty} \left| \frac{\dfrac{(-1)^{k+1}x^{3(k+1)+1}}{2^{k+1}(k+2)}}{\dfrac{(-1)^k x^{3k+1}}{2^k(k+1)}} \right| = \lim_{k\to\infty} \frac{|x|^3(k+1)}{2(k+2)} = \frac{|x|^3}{2}.$$

Thus the series converges if $|x|^3/2 < 1$ and diverges if $|x|^3/2 > 1$. Therefore

$$\text{Radius convergence} = \sqrt[3]{2}.$$

□

(b) $\displaystyle\sum_{k=0}^{\infty} k4^k(x-3)^k$.

Solution: This time we use the root test.

$$\text{root} = \lim_{k\to\infty} \left|k4^k(x-3)^k\right|^{\frac{1}{k}} = \lim_{k\to\infty} k^{\frac{1}{k}} 4|x-3| = 4|x-3|.$$

Whence the series converges if $4|x-3| < 1$ and diverges if $4|x-3| > 1$. Therefore

$$\text{Radius convergence} = \frac{1}{4}.$$

□

2. (a) Define what it means for the series $\displaystyle\sum_{k=1}^{\infty} a_k$ to be ***absolutely convergent***.

Solution: This means that the series $\displaystyle\sum_{k=1}^{\infty} |a_k|$ of absolute values converges. □

(b) Define what it means for the series $\displaystyle\sum_{k=1}^{\infty} a_k$ to be ***conditionally convergent***. □

Solution: This means that the original series $\displaystyle\sum_{k=1}^{\infty} a_k$ converges, but the series of absolute values $\displaystyle\sum_{k=1}^{\infty} |a_k|$ diverges.

(c) State the ***alternating series test***.

Solution: Let $a_1, a_2, a_3, \ldots$ be a series of numbers with

$$a_1 \ge a_2 \ge a_3 \ge a_4 \ge \cdots \quad \text{and} \quad \lim_{k\to\infty} a_k = 0$$

(that is the sequence decreases to 0). Then the alternating series

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + x_3 - a_4 + a_5 - a_6 + \cdots$$

converges.

And while this part we not required to get the problem correct, we even know a bit more. If $A_n = \sum_{k=1}^{n}(-1)^k a_k$ is the n-th partial sum and $A = \sum_{k=1}^{\infty}(-1)^k a_k$ is the sum of the series, then

$$|A - A_n| \leq a_{n+1}.$$

That is by stopping at the n-th partial sum, we only make an error of at most the size $(n+1)$-st term. □

(d) Give an example of a conditionally convergent series.

Solution: The standard example is the alternating harmonic series

$$\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

This converges by the alternating series test, but the series of absolute values $\sum_{k=1}^{\infty}\frac{1}{k}$ is a divergent series. □

3. Let $\displaystyle\sum_{k=1}^{\infty} a_k$ be a series of positive terms. State and prove the root test for this series.

Solution: If the limit

$$\rho = \lim_{k\to\infty}(a_k)^{\frac{1}{k}}$$

exits, then $\rho < 1$ implies the series converges and $\rho > 1$ implies the series diverges.

Proof: If $\rho < 1$ then choose any r with $\rho < r < 1$. Then as $\lim_{k\to\infty}(a_k)^{\frac{1}{k}} = \rho < r$ there is a N such that

$$n > N \quad \Longrightarrow \quad (a_k)^{\frac{1}{k}} < r.$$

But then $k > N$ implies

$$a_k < r^k$$

and so $\sum_{k=1}^{\infty} a_k$ converges by comparison with the convergent geometric series $\sum_{k=1}^{\infty} r^k$.

If $\rho > 0$ then there is a N such that

$$k > N \quad \Longrightarrow \quad (a_k)^{\frac{1}{k}} > 1.$$

This implies that $a_k > 1$ for $k > N$ and thus that

$$\lim_{k\to\infty} \neq 0.$$

This implies the series diverges. □

4. (a) Give a reasonably exact statement of the theorem on integrating power series term by term.

Solution: If the power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

has radius of convergence $r > 0$, the for all x with $|x| < r$ the integral

$$\int_0^x f(t)\,dt$$

exsits and it given by the termwise integration of the power series, that is

$$\int_0^x f(t)\,dt = \sum_{k=0}^{\infty} \frac{a_k}{k+1} x^{k+1} = \sum_{k=1}^{\infty} \frac{a_{k-1}}{k} x^k$$

□

(b) The function e^{-x^2} has the power series

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{k!}.$$

Find the power series for the function

$$f(x) = \int_0^x e^{-t^2}\,dt.$$

Solution: Just intergate termwise to get

$$f(x) = \int_0^x e^{-t^2}\,dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}.$$

□

5. Complete the following:

Theorem. Let $f\colon [a,b] \to \mathbb{R}$ be a real valued function on $[a,b]$. Then f is Riemann integrable on $[a,b]$ if and only if for all $\varepsilon > 0$ there are step functions ϕ and ψ such that ...

the inqualities

$$\phi(x) \le f(x) \le \psi(x)$$

hold on $[a,b]$ and

$$\int_a^b (\psi - \phi)\,dx < \varepsilon.$$

□

6. Define a function f on the real numbers by

$$f(x) = \sum_{k=1}^{\infty} \frac{\cos(4^k x)}{2^k}.$$

(a) Explain briefly (you mostly have to quote the correct theorem(s)) why f is continuous (you may assume that $\cos(x)$ is continuous).

Solution: The series converges uniformly (by comparison with the series of constants $\sum_{k=1}^{\infty} \frac{1}{2^k}$). Thus f is the uniform limit of the partial sums and the partial sums are continuous. As the uniform limit of continuous functions is continuous, we see that f is continuous. □

(b) Explain why we know that $\int_0^{\pi} f(x)\,dx$ exits. (Do not make this hard, it is just a sentence or two using part (a)).

Solution: By part (a) the function $f(x)$ is continuous. And we know that continuous functions are Riemannian integrable. ☐

(c) Show $\int_0^\pi f(x)\,dx = 0$.

Solution: Formally we have

$$\int_0^\pi f(x)\,dx = \int_0^\pi \left(\sum_{k=1}^\infty \frac{\cos(4^k x)}{2^k} \right) dx = \sum_{k=1}^\infty \int_0^\pi \frac{\cos(4^k x)}{2^k}\,dx = \sum_{k=1}^\infty 0 = 0.$$

This works as we can pass the integral inside the sum as the series is uniformly convergent. ☐