Mathematics 555 Test #1 Name: Solution Key

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1. (a) State the ***Weierstrass*** M ***test***.

Solution: Let X be a set and $f_1, f_2, f_3 \cdots X \to \mathbb{R}$ be a sequence of functions. Assume that there are constants $\langle K_k \rangle_{k=1}^{\infty}$ such that

(i) $|f_k(x)| \le M_k$ for all $x \in X$, and

(ii) $\displaystyle\sum_{k=1}^{\infty} M_k < \infty$.

Then the series $\displaystyle\sum_{k=1}^{\infty} f_k(x)$ converges uniformly and absolutely. □

(b) Let $f \colon \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{(x-k)^2 + k^2}.$$

Prove f is continuous.

Each of the functions $f_k(x) := \dfrac{1}{(x-k)^2 + k^2}$ is continuous and we have a theorem that tells us that the sum of uniformly convergent seres of continuous functions is continuous. So it is enough to show that the series defining f is uniformly convergent. We do this using the Weierstrass M test. Note

$$|f_k(x)| = \frac{1}{(x-k)^2 + k^2} \le \frac{1}{0 + k^2} = \frac{1}{k^2} =: M_k$$

And for this choice of M_k we have

$$\sum_{k=1}^{\infty} M_k = \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

as $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is p-series with $p > 1$. Therefore the series defining f is absolutely and uniformly continuous which completes the proof. □

2. (a) Let I be an interval in $\mathbb{R}$ and $f \colon I \to \mathbb{R}$ a function. Define what it means for f to be convex.

Solution: The function f is continuous on I iff for all $x, y \in I$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$ the inequality

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y).$$

(Geometrically this means that f lies below all its secants.) □

(b) State ***Jensen's inequality***.

Solution: If f is convex on the interval I and $x_1, x_2, \ldots, x_n \in I$ and $\alpha_1, \alpha_2, \ldots, \alpha_n$ are positive numbers with

$$\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$$

then the inequality

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_n) + \alpha_2 f(x_n) + \cdots + \alpha_n f(x_n)$$

holds. $\square$

(c) What is the second derivative criterion for a function to be convex?

Solution: If f is a function on an open interval I which is twice differentiable on I and $f'' \geq 0$ on I, then f is convex on I. $\square$

(d) Show that $f(x) = e^x$ is convex Jensen's inequality and use this so show that for any positive real numbers a, b, c that

$$a^2 b^3 c^4 \leq \left(\frac{2a + 3b + 4c}{9}\right)^9.$$

Solution: As $f''(x) = e^x > 0$ on R we have that f is convex by the second derivative criterion for a function to be convex.

This for any $x_1, x_2, x_3 \in \mathbb{R}$ if we use $\alpha_1 = \frac{2}{9}, \alpha_2 = \frac{3}{9}, \alpha_3 = \frac{4}{9}$ in Jensen's inequality we have

$$\begin{aligned}
(e^{x_1})^{\frac{2}{9}}(e^{x_2})^{\frac{3}{9}}(e^{x_3})^{\frac{4}{9}} &= e^{\frac{2}{9}x_1 + \frac{3}{9}x_2 + \frac{4}{9}x_3} \\
&= f\left(\frac{2}{9}x_1 + \frac{3}{9}x_2 + \frac{4}{9}x_3\right) \\
&\leq \frac{2}{9}f(x_1) + \frac{3}{9}f(x_2) + \frac{4}{9}f(x_3) \\
&= \frac{2}{9}e^{x_1} + \frac{3}{9}e^{x_2} + \frac{4}{9}e^{x_3}
\end{aligned}$$

Now for any $a, b, c > 0$ there are $x_1, x_2, x_3 \in \mathbb{R}$ with $a = x_1, b = x_2, c = x_3$. Using this in the last inequality gives

$$\left(a^2 b^3 c^4\right)^{\frac{1}{9}} = (e^{x_1})^{\frac{2}{9}}(e^{x_2})^{\frac{3}{9}}(e^{x_3})^{\frac{4}{9}} \leq \frac{2}{9}e^{x_1} + \frac{3}{9}e^{x_2} + \frac{4}{9}e^{x_3} = \frac{2a + 3b + 4c}{9}$$

Raising this to the 9-th power gives the desired inequality. $\square$

3. (a) State the ***Weierstrass Approximation Theorem.***

Solution: Let f be a continuous function on a closed interval $[a, b]$. Then there is a sequence of polynomials $\langle p_n \rangle_{n=1}^{\infty}$ such that

$$\lim_{n \to \infty} p_n(x) = f(x)$$

uniformly on $[a, b]$. $\square$

Alternate solution: Let f be a continuous function on a closed interval $[a, b]$ and let $\varepsilon > 0$. Then there is a polynomial $p(x)$ such that

$$|f(x) - p(x)| < \varepsilon$$

for all $x \in [a, b]$. $\square$

(b) Let f be a continuous function on $[0, 1]$ such that for all $n = 0, 1, 2, 3, \ldots$

$$\int_0^1 x^n f(x)\, dx = \frac{1}{n+3}.$$

Show that $f(x) \equiv x^2$. *Hint:* If $g(x) = f(x) - x^2$ what is $\int_0^1 g(x)x^n \, dx$?

Solution: As per the hint:

$$\int_0^1 g(x)x^n \, dx = \int_0^1 \left(f(x) - x^2 -\right) x^n \, dx = \int_0^1 f(x)x^n \, dx - \int_0^1 x^{n+2} \, dx = \frac{1}{n+3} - \frac{1}{n+3} = 0.$$

As we have seen in class several times this fact, along with linearity of the integral, implies

$$\int_0^1 g(x)p(x) \, dx = 0$$

for all polynomials $p(x)$.

The function g is continuous on $[0,1]$ therefore there is a sequence of polynomial $p_1, p_2, p_3, \ldots$ such that

$$\lim_{n\to\infty} p_n(x) = g(x)$$

uniformly on $[0,1]$. But this we also have

$$\lim_{n\to\infty} g(x)p_n(x) = g(x)g(x) = g(x)^2$$

uniformly on $[0,1]$. Therefore by one of our basic limit theorem

$$0 = \lim_{n\to 0} 0 = \lim_{n\to\infty} \int_0^1 g(x)p_n(x) \, dx = \int_0^1 g(x)^2 \, dx.$$

As $g(x)^2 \geq 0$ and g^2 is continuous this implies $g(x) \equiv 0$. This implies $f(x) \equiv x^2$ on $[0,1]$. $\square$

4. (a) Give our official (that is the series) definition of $\cos(x)$ and $\sin(x)$.

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$
$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

$\square$

(b) Show that if $g''(x) = -g(x)$ for all x and $g(0) = g'(0) = 0$ then $g \equiv 0$.

Solution: Let $E = g^2 + (g')^2$. Then

$$E' = 2gg' + 2g'g'' = 2gg' + 2g'(-g) = 0.$$

Therefore E is constant. Letting $t = 0$ we find

$$E = g(0)^2 + g'(0)^2 = 0^2 + 0^2 = 0.$$

Thus $g(x)^2 + g'(x)^2 \equiv 0$. This implies $g(x) \equiv 0$. $\square$

(c) Show that if $f''(x) = -f(x)$ and for all x, then $f(x) = f(0)\cos(x) + f'(0)\sin(x)$. (You may assume that $\cos''(x) = -\cos(x)$ and $\sin''(x) = -\sin(x)$.)

Solution: Let

$$g(x) = f(x) - f(0)\cos(x) - f'(0)\sin(x).$$

Then, using $f'' = -f$,

$$\begin{aligned} g'(x) &= f'(x) + f(0)\sin(x) - f'(0)\cos(x), \\ g''(x) &= f''(x) + f(0)\cos(x) + f'(0)\sin(x) = -f(x) + f(0)\cos(x) + f'(0)\sin(x) = -g(x). \end{aligned}$$

Also

$$\begin{aligned} g(0) &= f(0) - f(0)\cos(0) - f'(0)\sin(0) = f(0) - f(0)1 - f'(0)0 = 0 \\ g'(0) &= f'(0) + f'(x) + f(0)\sin(0) - f'(0)\cos(0) = f'(0) + f(0)0 - f'(0)1 = 0. \end{aligned}$$

That is $g'' = -g$ and $g(0) = 0$ $g'(0) = 0$. So by part (b) $g(x) \equiv 0$, which implies $f(x) = f(0)\cos(x) + f'(0)\sin(x)$ as required. $\square$

5. (a) Define that it means for $\langle K_n(x)\rangle_{n=1}^{\infty}$ to be a ***Dirac sequence***.

Solution: Each K_n is a Riemannian integrable function and the following three conditions hold:

(i) $K_n(x) \geq 0$ for all x,

(ii) $\displaystyle\int_{-\infty}^{\infty} K_n(x)\,dx = 1$, and

(iii) For all $\delta > 0$ the limit $\displaystyle\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx = 0$ holds. $\square$

(b) Show that the functions

$$K_n(x) = \begin{cases} n, & n \leq x \in [0, 1/n]; \\ 0, & x \notin [0, 1/n] \end{cases}$$

form a Dirac sequence.

That condition (i) holds is clear. For condition (ii) not

$$\int_{-\infty}^{\infty} K_n(x)\,dx = \int_0^{1/n} n\,dx = n\frac{1}{n} = 1.$$

For (iii) given $\delta > 0$ we choose N such that $\frac{1}{n} < \delta$. Then if $|x| \geq \delta$ and $n > N$ we have $|x| \geq \delta \geq \frac{1}{N} > \frac{1}{n}$ and thus $K_n(x) = 0$. Whence

$$n > N \quad \Longrightarrow \quad \int_{|x|\geq\delta} K_n(x)\,dx = \int_{|x|\geq\delta} 0\,dx = 0.$$

This shows $\displaystyle\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx = 0$. $\square$

(c) We know that if f is a bounded continuous function on $\mathbb{R}$ and $\langle K_n(x)\rangle_{n=1}^{\infty}$ is a Dirac sequence, then the functions

$$f_n(x) := \int_{-\infty}^{\infty} K_n(x-y)f(y)\,dy$$

converge to $f(x)$ pointwise. In the case of the Dirac sequence of part (b) this follows form anther fundamental result. What is it? *Hint:* Explicitly write out $\int_{-\infty}^{\infty} K_n(x-y)f(y)\,dy$ this should become clear.

Solution: If $x-y=0$, then $y=x$ and if $x-y=1/n$ we have $y=x-1/n$. Thus by the definition of K_n we have that $K_n(x-y)=n$ for $x-1/n \le y \le x$ and is $K_n(x-y)=0$ for all other values of y. Thus

$$f_n(x)=\int_{-\infty}^{\infty} K_n(x-y)f(y)\,dy=\int_{x-1/n}^{x} nf(y)\,dy=n\int_{x-1/n}^{x} f(y)\,dy.$$

If we set

$$F(x)=\int_0^x f(y)\,dy$$

then

$$f_n(x)=n\int_{x-1/n}^{x} f(y)\,dy=\frac{F(x)-F(x-1/n)}{1/n}$$

and thus by the Fundamental Theorem of Calculus

$$lim_{n\to\infty}f_n(x)=\lim_{n\to\infty}\frac{F(x)-F(x-1/n)}{1/n}=\lim_{h\to 0}\frac{F(x)-F(x-h)}{h}=F'(x)=f(x).$$

□