Math 555 Homework

Here is anther fact about continuous functions we should know.

Theorem 1. *Let $f\colon [a,b] \to [c,d]$ be onto, continuous and strictly increasing (or strictly decreasing). Then the inverse $f^{-1}\colon [c,d] \to [a,b]$ is also continuous.*

Problem 1. Prove this. *Hint:* We are given that f is onto. As f is strictly increasing it is also one-to-one. This implies that the inverse f^{-1} exists. One of the ways to shot that function g is continuous is to show that g^{-1}[closed set] is a closed. In our case $g = f^{-1}$ and thus $g^{-1} = (f^{-1})^{-1} = f$. So we only need show that for any closed subset C of $[a,b]$ that $f[C]$ is a closed subset of $[c,d]$. As $[a,b]$ is compact (why?) we have that C is compact (why?). Therefore $f[C]$ is the continuous image of a compact set and thus $f[C]$ is a compact subset of $[c,d]$. Therefore $f[C]$ is compact (why?). Finally this implies that $f[C]$ is closed (why?) which finishes the proof. □

There is a generalization of this that has almost exactly the same proof.

Problem 2 (Extra Credit). If $f\colon E \to E'$ is a continuous one-to-one onto (or to be French about it f is bijective) between metric spaces with E compact. Then the inverse $f^{-1}\colon E' \to E$ is continuous. *Hint:* To start note that since f is onto we have $f[E] = E'$ and thus E' is the continuous image of a compact set and thus compact. □

Now let's get started with derivatives.

Definition 2. Let $f\colon U \to \mathbf{R}$ be a real valued function defined on an open subset U of $\mathbf{R}$. Then f is ***differentiable*** at $x_0 \in U$ iff the limit

$$f'(x_0) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exsits. □

The limit defining the derivative can also be written as

$$f'(x_0) = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We prove the following in class.

Theorem 3. *If f is differentiable at x_0, then f is continuous at x_0.*

Proof. By a result from last semester it is enough to show $\lim_{x\to x_0} f(x) = f(x_0)$.

$$\begin{aligned}\lim_{x\to x_0} f(x) &= \lim_{x\to x_0} \left(f(x_0) + \frac{f(x) - f(x_0)}{x - x_0}(x - x_0) \right) \\ &= f(x_0) + f'(x_0)\cdot 0 \\ &= f(x_0).\end{aligned}$$

□

We next verify all the usual rules for derivatives that we know and love.

Proposition 4. *Let $f, g\colon U \to \mathbf{R}$ be defined on an open set $U \subseteq \mathbf{R}$. If both f and g are differentiable at $x_0 \in U$, then so is $f+g$ and*

$$(f+g)'(x_0) = f'(x_0) + g'(x_0).$$

Problem 3. Prove this. □

Proposition 5. *Let $f\colon U \to \mathbf{R}$ be defined on an open set $U \subseteq \mathbf{R}$ and let $c \in \mathbf{R}$. If f is differentiable at $x_0 \in U$, then so is cf and*

$$(cf)'(x_0) = cf'(x_0).$$

Problem 4. Prove this. □

Proposition 6 (Product Rule.). *Let $f, g\colon U \to \mathbf{R}$ be defined on an open set $U \subseteq \mathbf{R}$. If both f and g are differentiable at $x_0 \in U$, then so is the product fg and*

$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$

Problem 5. Prove this. □

Proposition 7. *Let $g\colon U \to \mathbf{R}$ be defined on an open set $U \subseteq \mathbf{R}$ and let $c \in \mathbf{R}$. If g is differentiable at $x_0 \in U$, then so is $\frac{1}{g}$ and*

$$\left(\frac{1}{g}\right)'(x_0) = \frac{-g'(x_0)}{g(x_0)^2}.$$

Problem 6. Prove this. □

Proposition 8 (Quotient Rule). *Let $f, g\colon U \to \mathbf{R}$ be defined on an open set $U \subseteq \mathbf{R}$. If both f and g are differentiable at $x_0 \in U$ and $g(x_0) \neq 0$ then the quotient $\frac{f}{g}$ is also differentiable at x_0 and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Problem 7. Prove this. *Hint:* Combine Proposition 6 and Proposition 7. □

It is now time so show that some differentiable functions exist.

Proposition 9. *Let $f\colon \mathbf{R} \to \mathbf{R}$ be the function given by $f(x) = mx + b$ where m and b are constants. Then f differentiable at all points of $\mathbf{R}$ and*

$$f'(x) = m.$$

Problem 8. Prove this. □

Problem 9. From the last problem we know that $x' = 1$. Use this fact and the Product Rule to show that if $f(x) = x^2$ then f is differentiable at all points of $\mathbf{R}$ and $f'(x) = 2x$. □

Problem 10. Let n be a positive integer and let $f\colon \mathbf{R} \to \mathbf{R}$ be the function $f(x) = x^n$. Use induction and the product rule to show that f is differentable at all points of $\mathbf{R}$ and that $f'(x) = nx^{n-1}$. □