Math 554 Homework

In the last homework we proved

Theorem 1 (Cauchy Mean Value Theorem)**.** *Let f and g be continuous on $[a,b]$ and differentiable on (a,b). There there is a $\xi \in (a,b)$ such that*

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

(Note if $g(b) - g(a) \neq 0$ and $g'(\xi) \neq 0$ this and be written as

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(\xi)}{g'(\xi)}$$

which will be useful below.) □

We now wish to look at one of the other standard topics in differential calculus, l'hôpital's rule. Recall this involves evaluating limits of the type

$$\lim_{x \to x_0} \frac{f(x)}{g(x)}$$

where $f(x_0) = g(x_0) = 0$ which leads to

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{0}{0}$$

which, at least formally, does not make sense. Here is the basic result.

Theorem 2 (L'hôpital's rule)**.** *Let f and g be differentiable in a neighborhood of x_0 with $g'(x)] \neq 0$ for $x \neq x_0$. Assume that $f(x_0) = g(x_0) = 0$ and that*

$$\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$$

exists. Then $\lim_{x \to x_0} f(x)/g(x)$ exists and is given by

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = L$$

This is usually stated informally as that if $f(x_0) = g(x_0) = 0$ then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

The important part is that the existence of the limit $\lim_{x \to x_0} \frac{f'(x)}{g'(x)}$ implies the existence of the limit $\lim_{x \to x_0} \frac{f(x)}{g(x)}$.

Problem 1. Prove Theorem 2 as follows. Let $\varepsilon > 0$ then as $\lim_{x \to x_0} \frac{f'(x)}{g'(x)} = L$ there is a $\delta > 0$ such that

$$0 < |x - x_0| < \delta \quad \implies \quad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon.$$

Let x be so that $0 < |x - x_0| < \delta$. Then, by the Cauchy Mean Value Theorem, there is a ξ between x and x_0 such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - 0}{g(x) - 0} = \frac{f(x) - f(x_0)}{g(x) - g(x_0)} = \frac{f'(\xi)}{g'(\xi)}.$$

Use this to show

$$0 < |x - x_0| < \delta \quad \Longrightarrow \quad \left| \frac{f(x)}{g(x)} - L \right| < \varepsilon$$

and thus $\lim_{x\to x_0} \frac{f(x)}{g(x)} = L$. (A main point is that $0 < |\xi - x_0| < \delta$, so be sure to explain why this holds.) □

Here is a standard application of l'hôpital's rule:

$$\lim_{x\to 0} \frac{\sin(2x)}{3x} = \lim_{x\to 0} \frac{\sin(2x)'}{(3x)'} = \lim_{x\to 0} \frac{2\cos(2x)}{3} = \frac{2\cos(0)}{3} = \frac{2}{3}.$$

It can also be applied several times in a row:

$$\begin{aligned} \lim_{x\to 0} \frac{1 - \cos(x)}{x^2} &= \lim_{x\to 0} \frac{\sin(x)}{2x} && \left(\text{take } \frac{d}{dx} \text{ of top and bottom}\right) \\ &= \lim_{x\to 0} \frac{\cos(x)}{2} && \left(\text{take } \frac{d}{dx} \text{ of top and bottom}\right) \\ &= \frac{\cos(0)}{2} \\ &= \frac{1}{2}. \end{aligned}$$

So we have shown $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \frac{1}{2}$. Note that in terms of showing this limit exists, this should be read from the bottom up. That is l'hôpital's rule shows that $\lim_{x\to 0} \frac{\sin(x)}{2x}$ exists as $\lim_{x\to 0} = \frac{\sin(x)'}{(2x)'} = \lim_{x\to 0} \frac{\cos(0)}{2} = \frac{1}{2}$ exists. Then anther application of l'hôpital's rule shows that $\lim_{x\to 0} \frac{1-\cos(x)}{x^2} = \lim_{x\to 0} \frac{(1-\cos(x))'}{(x^2)'} = \lim_{x\to 0} \frac{\sin(x)}{2x}$ exists.

Problem 2. Here are some problems to practice the use of l'hôpital's rule. Compute the following

(a) $\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$

(b) $\lim_{x\to 0} \frac{e^x - e^{-x}}{x}$

(c) $\lim_{\theta\to \pi} \frac{\sin^3(x)}{x(\cos(x) + 1)}$

□

Now back to Rôlle's theorem. First a definition.

Definition 3. Let f be defined on an open integral I. Then f is ***twice differentiable*** on I if f' exsits at all points of I and the function f' is differentiable on I. We denote the derivative of f' as f'' or $f^{(2)}$ and it is called the ***second derivative*** of f. If f'' exists at all points of I and f'' is differentiable on I its derivative is denoted by f''' or $f^{(3)}$ and is called the ***third derivative*** of f and f is said to be ***three times differentiable***. Continuing recursively, if we have defined what it means for f to be n ***times differentiable*** on I and the ***n-th derivative***, $f^{(n)}$, is differentiable on I then the derivative of $f^{(n)}$ is denoted by $f^{(n+1)}$ and f is $(n+1)$ ***times differentiable*** on I. □

Remark. For consistency's sake we set $f^{(0)} = f$ and $f^{(1)} = f'$ □

Problem 3. Show that the function f on $\mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2, & x \geq 0; \\ -x^2, & x < 0. \end{cases}$$

is differentiable on $\mathbb{R}$ but not twice differentiable. *Hint:* Show $f'(x) = 2|x|$. You may have to use the limit definition to compute $f'(0)$. □

Problem 4. Find a function that is twice differentiable on $\mathbb{R}$ but not three times differentiable. More generally can you give an example of a function that is n times differentiable, but not $n+1$ times differentiable. □

Proposition 4. *Let I be an open interval and assume f is twice differentiable on I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume $f(x_0) = f'(x_0) = 0$ and $f(x_1) = 0$. Then there is a point ξ between x_0 and x_1 with $f''(\xi) = 0$.*

Proof. As $f(x_0) = f(x_1) = 0$ by Rôlle's Theorem there is a ξ_1 between x_0 and x_1 with $f'(\xi_1) = 0$. But the function f' is differentiable on I and $f'(x_0) = f'(\xi_1) = 0$ and thus anther application of Rôlle's Theorem gives us a ξ between x_0 and ξ_1 with $f''(\xi) = (f')'(\xi) = 0$. As ξ_1 is between x_0 and x_1 and ξ is between x_0 and ξ_1 we have that ξ is between x_0 and x_1. □

This generalizes

Theorem 5. *Let f be $n+1$ times differentiable on the open interval I. Let $x_0, x_1 \in I$ with $x_0 \neq x_1$. Assume that*

- $f(x_0) = f'(x_0) = \cdots = f^{(n)}(x_0) = 0,$
- $f(x_1) = 0.$

Then there is a point ξ between x_0 and x_1 with

$$f^{(n+1)}(\xi) = 0.$$

□

Problem 5. Prove this. *Hint:* There are several ways to do this. One is to look at the proof of Proposition 4 and meditate upon induction. □

Proposition 6. *Let f be twice differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b with*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2.$$

Proof. Let h be defined on I by

$$h(x) = f(x) - f(a) - f'(a)(x-a) - c(x-a)^2$$

where c is a constant to be chosen shortly. Note

$$h(a) = 0$$

and

$$h'(x) = f'(x) - f'(a) - 2c(x-a),$$

and thus

$$h'(a) = 0.$$

With applying Proposition 4 in mind, we choose c so that $h(b) = 0$. That is

$$h(b) = f(b) - f(a) - f'(a)(b-a) - c(b-a)^2 = 0$$

which leads to

$$c = \frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}.$$

With this choice of c we have $h(a) = h'(a) = h(b) = 0$ and thus by Proposition 6 there is a ξ between a and b with

$$h''(\xi) = 0.$$

By direct calculation

$$h''(x) = f''(x) - 2c.$$

Then $h''(\xi) = 0$ yields

$$f''(\xi) - 2c = 0.$$

But using the formula for c above we find

$$f''(\xi) - 2\left(\frac{f(b) - f(a) - f'(a)(b-a)}{(b-a)^2}\right) = 0$$

which can be rearranged to give

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(\xi)}{2}(b-a)^2$$

as required. □

As this was a more or less direct consequence of Proposition 4 it makes sense to look for a generalization that depends on Theorem 5. To make life a little easier on ourselves we first do the case of $n = 4$.

Lemma 7. *Let f be a function that is four times differentiable on an open interval I and let $a \in I$. Let $T(x)$ be the polynomial*

(1)

$$T(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4,$$

and set

$$g(x) = f(x) - T(x).$$

Then

$$g(a) = g'(a) = g''(a) = g^{(3)}(a) = g^{(4)}(a) = 0.$$

Problem 6. Prove this. ☐

Theorem 8. *Let f be five times differentiable on the open interval I and $a, b \in I$ with $a \neq b$. Then there is a ξ between a and b such that*

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2 + \frac{f^{(3)}(a)}{3!}(b-a)^3 + \frac{f^{(4)}(a)}{4!}(b-a)^4 + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Or in different notation let $T(x)$ be the polynomial (1), *then this is*

$$f(b) = T(b) + \frac{f^{(5)}(\xi)}{5!}(b-a)^5.$$

Problem 7. Prove this. *Hint:* Let

$$h(x) = f(x) - T(x) - c(x-a)^5$$

where we choose c so that

$$h(b) = 0.$$

Show $h(a) = h'(a) = h''(a) = h^{(3)}(a) = h^{(4)}(a) = 0$. Now use Theorem 5 and now proceed as in the proof of Proposition 6. ☐

Definition 9. Let f be n times differentiable on a neighborhood of a. Then the ***degree*** n ***Taylor polynomial*** of f at x is

$$T_n(x) := \sum_{k=0}^{n} f^{(k)}(a)\frac{(x-a)^k}{k!}.$$

☐

Problem 8. Show that if f is n times differentiable on an open interval I and T_n is its degree n Taylor polynomial at a, then for $0 \leq k \leq n$

$$T_n^{(k)}(a) = f^{(k)}(a).$$

That is the k-th derivatives of T_n and f agree at a for $0 \leq k \leq n$. ☐

Theorem 10 (Taylor's Theorem with Lagrange's form of the remainder)**.** *Let f be $(n+1)$ times differentiable on the open interval I and let $a, b \in I$ with $a \neq b$. Let T_n be the degree n Taylor polynomial of f at a. Then there is a ξ between a and b such that*

$$f(b) = T_n(b) + f^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!}.$$

(The term $E_n(b) = f^{(n+1)}(\xi)\frac{(b-a)^{n+1}}{(n+1)!} = f(b) - T_n(b)$ is the ***error term*** *or* ***remainder term*** *when approximating f by its Taylor polynomial T_n.)*

Problem 9. Prove this. ☐

We restate this with slightly different notation (just replacing a and b with x_0 and x.)

Theorem 11 (Taylor's Theorem with Lagrange's form of the remainder, form 2)**.** *Let f be $(n+1)$ times differentiable on the open interval I and let $x, x_0 \in I$ with $x \neq x_0$. There there is a ξ betweem x and x_0 such that*

$$f(x) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{(x-x_0)^k}{k!} + f^{(n+1)}(\xi)\frac{(x-x_0)^{n+1}}{(n+1)!}. \qquad \square$$

Remark. In the case that $n = 0$ this becomes

$$f(x) = f(x_0) + f'(\xi)(x - x_0),$$

which can be rewritten as $f(x) - f(x_0) = f'(\xi)(x - x_0)$. That is for $n = 0$ we just get the mean value theorem. $\square$

One last restatement of Taylor's theorem. If we let $x = x_0 + h$ we get

$$f(x_0 + h) = \sum_{k=0}^{n} f^{(k)}(x_0)\frac{h^k}{k!} + f^{(n+1)}(\xi)\frac{h^{n+1}}{(n+1)!}$$

where ξ is between x_0 and $x_0 + h$.

As an examples of Taylor's theorem we have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{e^\xi x^4}{4!} \qquad \text{(Used } n = 3.\text{)}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{\cos(\xi)x^6}{6!} \qquad \text{(Used } n = 5.\text{)}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{\cos(\xi)x^7}{7!} \qquad \text{(Used } n = 6.\text{)}$$

where ξ is between x and 0 (and of course the value of ξ is different in each of the three equations).