Math 554 Homework

1. Existence results for the Riemannian integral.

Here is a theorem we missed because of the snow days.

Theorem 1. *The function f defined on $[a,b]$ is Riemann integrable if and only if for all $\varepsilon > 0$ there are step functions ϕ and ψ with*

$$\phi \leq f \leq \psi$$

on $[a,b]$ and

$$\int_a^b (\psi(x) - \phi(x))\, dx \leq \varepsilon.$$

□

We are not going to go back and prove this, but here are some applications. First a preliminary result.

Lemma 2. *Let $f\colon [a,b] \to \mathbb{R}$ be continuous and $\varepsilon > 0$. Then there is a partition $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ such that for each interval $[x_{j-1}, x_j]$ if $x, y \in [x_{j-1}, x_j]$, then*

$$|f(x) - f(y)| \leq \frac{\varepsilon}{b-a}$$

Problem 1. Prove this. *Hint:* As f is continuous on the compact set $[a,b]$ it is uniformly continuous. Therefore there is a $\delta > 0$ such that for all $x, y \in [a,b]$

$$|x - y| \leq \delta \quad \implies \quad |f(x) - f(y)| \leq \frac{\varepsilon}{b-a}.$$

Now choose a partition that is δ fine and show it works. □

Theorem 3. *If $f\colon [a,b] \to \mathbb{R}$ is continuous, then it is Riemann integrable.*

Problem 2. Prove this. *Hint:* Let $\varepsilon > 0$. Choose a partition as in Lemma 2 and for each j with $1 \leq j \leq n$ set

$$M_j := \max\{f(x) : x_{j-1} \leq x \leq x_j\}, \qquad m_j = \min\{f(x) : x_{j-1} \leq x \leq x_j\}.$$

As $[x_{j-1}, x_j]$ is compact the maximums and minimums exist and there are $\xi_j, \eta_j \in [x_{j-1}, x_j]$ with

$$f(\xi_j) = m_j, \qquad f(\eta_j) = M_j.$$

Let ϕ and ψ be the step functions

$$\phi(x) = \begin{cases} m_j, & x_{j-1} \leq x < x_j \ \text{ and } 1 \leq j \leq n; \\ f(b), & x = x_n = b. \end{cases}$$

and

$$\psi(x) = \begin{cases} M_j, & x_{j-1} \leq x < x_j \ \text{ and } j < n; \\ f(b), & x = x_n = b. \end{cases}$$

and show

$$\phi \le f \le \psi$$

on $[a, b]$ and

$$\int_a^b (\psi(x) - \phi(x))\, dx < \varepsilon.$$

Now use Theorem 1. □

Lemma 4. *Let* $\alpha, \beta \in \mathbb{R}$. *Then*

$$|\max\{\alpha, 0\} - \max\{\beta, 0\}| \le |\alpha - \beta|.$$

Problem 3. Prove this. □

Proposition 5. *Let* f *be Riemann integrable on* $[a, b]$. *Then the function*

$$g = \max\{f, 0\}$$

is also Riemann integrable on $[a, b]$.

Problem 4. Prove this. *Hint:* Let $\varepsilon > 0$. By Theorem 1 there are step functions ϕ and ψ such that

$$\phi \le f \le \psi$$

on $[a, b]$ and

$$\int_a^b (\psi(x) - \phi(x))\, dx \le \varepsilon.$$

Then explain briefly why

$$\phi_0(x) := \max\{\phi(x), 0\}, \qquad \psi_0(x) := \max\{\psi(x), 0\}$$

are step functions with

$$\phi_0 \le \max\{f, 0\} \le \psi_0$$

and

$$\int_a^b (\psi_0(x) - \phi_0(x))\, dx \le \int_a^b (\psi(x) - \phi(x))\, dx \le \varepsilon$$

and why this is enough to finish the proof. □

Lemma 6. *If* $\alpha\beta \in \mathbb{R}$, *then*

$$\begin{aligned} \min\{\alpha, 0\} &= -\max\{-\alpha, 0\} \\ |\alpha| &= \max\{\alpha, 0\} + \max\{-\alpha, 0\} \\ \max\{\alpha, \beta\} &= \alpha + \max\{0, \beta - \alpha\} \\ \min\{\alpha, \beta\} &= \alpha + \min\{0, \beta - \alpha\} \end{aligned}$$

Problem 5. Convince yourself this is true. I will not collect this problem, but be prepared to present it in class. □

Proposition 7. *If* f *and* g *are integrable on* $[a, b]$, *then so are* $|f|$, $\max\{f, g\}$, *and* $\min\{f, g\}$.

Problem 6. Use Proposition 5 and Lemma 6 to prove this. □

Proposition 8. *If f is integrable on $[a, b]$, then so is f^2.*

Problem 7. Prove this. *Hint:* As $f^2 = |f|^2$ and $|f|$ is also integrable by Proposition 7 we can relace f by $|f|$ and assume that $f \geq 0$. As f is integrable it is bounded and therefore there is a $B > 0$ such that $0 \leq f \leq B$. By Theorem 1 there are step functions ϕ and ψ with

$$\phi \leq f \leq \psi$$

on $[a, b]$ and

$$\int_a^b (\psi(x) - \phi(x))\, dx < \frac{\varepsilon}{2B}.$$

By replacing ϕ with $\max\{0, \phi\}$ and ψ by $\min\{B, \psi\}$ we can assume

$$0 \leq \phi \leq f \leq \psi \leq B.$$

Then ϕ^2 and ψ^2 are step functions and

$$\phi^2 \leq f^2 \leq \psi^2$$

on $[a, b]$. Also

$$\int_a^b (\psi^2(x) - \phi^2(x))\, dx = \int_a^b (\psi(x) + \phi(x))(\psi(x) - \phi(x)) dx.$$

Use this to show

$$\int_a^b (\psi^2(x) - \phi^2(x))\, dx \leq \varepsilon$$

and thus compete the proof. □

Theorem 9. *Let f and g be integrable on the interval $[a, b]$. Then the product fg is also integrable on $[a, b]$.*

Problem 8. Prove this. *Hint:* Yet anther artfully complication trick. First show

$$fg = \frac{(f + g)^2 - (f - g)^2}{4}.$$

Now it shoud be easy to complete the proof using Proposition 8. □

2. More on the Fundamental Theorem of Calculus and its Consequences.

We have proven the following

Theorem 10 (Fundamental Theorem of Calculus Form 1)**.** *Let f be Riemann integrable on $[a, b]$ and define $F\colon [a, b] \to \mathbb{R}$ by*

$$F(x) = \int_a^x f(t)\, dt.$$

Then at any point $x \in (a, b)$ where f is continuous the function F is continuous at x and

$$F'(x) = f(x).$$

□

The form of the Fundamental Theorem that is used most often in evaluating integrals is

Theorem 11 (Fundamental Theorem of Calculus Form 2)**.** *Let f be continuous on $[a,b]$ and let F be an **antiderivative** for f on $[a,b]$. (That is F is continuous on $[a,b]$ and $F'(x) = f(x)$ for all $x \in (a,b)$.) Then*

$$\int_a^b (x)\,dx = F(x)\Big|_a^b = F(b) - F(a).$$

Problem 9. Prove this. *Hint:* One way to start is to let $G\colon [a,b] \to \mathbb{R}$ be

$$G(x) = \int_a^x f(t)\,dt$$

and show that $(F-G)' = 0$ on (a,b) and thus $F-G$ is constant on $[a,b]$. □

Theorem 11 is what lets us do calculations we know and love such as

$$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = \frac{2^4 - 0^4}{4} = 4.$$

Theorem 12 (Integration by Parts)**.** *Let u, v be continuous on $[a,b]$ and differentable on (a,b) and assume that u' and v' are differentiable on (a,b). Then*

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big|_a^b - \int_a^b u'(x)v(x)\,dx.$$

Problem 10. Prove this. *Hint:* This follows form the product rule and the Fundamental Theorem of Calculus in the form

$$\int_a^b (u(x)v(x))'\,dx = u(x)v(x)\Big|_a^b.$$

You do have to worry about the existance of the integrals involved, but Theorem 3 and Theorem 9 should take care of this. □

Here is another consequence of the Fundamental Theorem of Calculus that you are use to using to evaluate integrals.

Theorem 13 (Change of Variable Formula)**.** *Let the map $x = u(t)$ map the interval $[c,d]$ into the interval $[a,b]$ and assume that $u'(t)$ is continuous on $[c,d]$. Then for any integrable function f on $[a,b]$*

$$\int_{u(c)}^{u(d)} f(x)\,dx = \int_c^d f(u(t))u'(t)\,dt.$$

Problem 11. Prove this. *Hint:* Do this in steps

(a) Explain why both the integrals exist.
(b) Define F on $[a,b]$ by

$$F(x) = \int_a^x f(y)\,dy$$

and explain why

$$F'(x) = f(x) \quad \text{and} \quad \int_{u(c)}^{u(d)} f(x)\,dx = F(u(d)) - F(u(c)).$$

(c) On $[c, d]$ define

$$G(t) = F(u(t)).$$

By the chain rule

$$G'(t) = F'(u(t))u'(t) = f(u(t))u'(t)$$

and so by Theorem 11

$$\int_c^d f(u(t))u'(t)\,dt = \int_c^d G'(t)\,dt = G(d) - G(c).$$

(d) Put the pieces above together to finish the proof. □

2.1. **More on Taylor's Theorem.** We now use integration by parts to give another form of the remainder in Taylor's Theorem.

Lemma 14. *Let f be $k+1$ times differentable on an open interval (α, β) and assume that $f^{(k+1)}$ is integrable. Then for $a, x \in (\alpha, \beta)$ we have*

$$\int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t)\,dt = \frac{f^{(k)}(a)}{k!}(x-a)^k + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t)\,dt.$$

Problem 12. Prove this. *Hint:* Use integration by parts with $v'(t) = \frac{(x-t)^{k-1}}{(k-1)!}$ and $u = f^{(k)}(t)$. □

Theorem 15 (Taylor's Theorem with Integral form of the Remainder)**.** *Let f be $n+1$ times differentable on (α, β) and assume that $f^{(n+1)}$ is integrable. Then for $a, x \in (\alpha, \beta)$*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder term $R_n(x)$ is given by

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t)\,dt.$$

Problem 13. Prove this. *Hint:* Note that Lemma 14 can be rewritten as

$$R_{k-1}(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

and by the Fundamental Theorem of Calculus and integration by parts

$$\begin{aligned} f(x) - f(a) &= \int_a^x f'(t)\,dt \\ &= -\int_a^x (-1)f'(t)\,dt \\ &= -\int_a^x \left(\frac{d}{dt}(x-t)\right) f'(t)\,dt \\ &= -\frac{d}{dt}(x-t)f'(t)\Big|_{t=a}^{x} + \int_a^x (x-t)f''(t)\,dt \\ &= f(a)(x-a) + R_1(x). \end{aligned}$$

Now use induction. □

3. The Logarithm and Exponential.

Define a function $L\colon (0,\infty) \to \mathbb{R}$ by

$$L(x) = \int_1^x \frac{dt}{t}.$$

We know this should be the natural logarithm, but we now verify directly from its definition that it has the correct properties.

Proposition 16. *The derivative of L is*

$$L'(x) = \frac{1}{x}$$

and thus L is strictly increasing. Therefore L is one-to-one (that is injective).

Proof. By the Fundamental Theorem of Calculus

$$L'(x) = \frac{1}{x} > 0$$

as $x > 0$ which implies L is strictly increasing. □

Proposition 17. *Let $a, b > 0$ then*

$$\int_a^b \frac{dx}{x} = L(b/a).$$

Problem 14. Prove this. *Hint:* In the integral $\int_a^b \frac{dx}{x}$ do the change of variable $x = at$ to get

$$\int_a^b \frac{dx}{x} = \int_1^{b/a} \frac{dt}{t}.$$ □

Proposition 18. *If $a, b > 0$ then*

$$L(ab) = L(a) + L(b).$$

Problem 15. Prove this. *Hint:*

$$L(ab) = \int_1^{ab} \frac{dx}{x} = \int_1^{a} \frac{dx}{x} + \int_a^{ab} \frac{dx}{x}$$

and use Proposition 17. □

The last Proposition and induction yield:

Corollary 19. *If* $a > 0$ *and* n *is a positive integer*

$$L(a^n) = nL(a).$$

□

Proposition 20. *The function* $L\colon (0,\infty) \to \mathbb{R}$ *is a bijection.*

Problem 16. Prove this. *Hint:* Recall the saying that L is a bijection is just saying that it is one-to-one and onto. We have already seen that L is injective. To see that it is surjective (that is onto) note that $L(2) > 0$ and $L(1/2) < 0$. Also for a positive integer n

$$L(2^n) = nL(2) \qquad \text{and} \qquad L(1/2^n) = nL(1/2).$$

If y_0 is any real number we can find (by Archimedes' principle) a positive integer n such that

$$nL(1/2) < y_0 < nL(2).$$

Also we know that L is continuous (why?). Now you should be able to show that there is a $x_0 \in (0,\infty)$ with $L(x_0) = y_0$. □

Because the function $L\colon (0,\infty) \to \mathbb{R}$ is bijective, it has an inverse $E\colon \mathbb{R} \to (0,\infty)$. As L is strictly increasing, continuous, and differentiable with $L'(x) \neq 0$ for all x we have theorems which tell us that E is strictly increasing, continuous, and differentiable.

Proposition 21. *The function* E *satisfies* $E(0) = 1$ *and*

$$E'(x) = E(x).$$

Problem 17. Prove this. *Hint:* $L(1) = 0$. And as L and E are inverses of each other $L(E(x)) = x$ for all $x \in \mathbb{R}$. Therefore $\frac{d}{dx}L(E(x)) = 1$. Use the chain rule and that we know the derivative of L. □

Proposition 22. *For all real numbers* x

$$E(-x) = \frac{1}{E(x)}.$$

Problem 18. Prove this. *Hint:* There are several ways to do this. One is to take the derivative of $E(x)E(-x)$ and show it is zero. Anther is to note that $L(a) + L(1/a) = L(1) = 0$ □

Proposition 23. *For all real numbers* a, b

$$E(a+b) = E(a)E(b).$$

Problem 19. Prove this. *Hint:* One way is to deduce this from the property $L(\alpha\beta) = L(\alpha) + L(\beta)$ of L. Anther is to show that the derivative of the function

$$f(x) = E(x + a)E(-x)$$

is zero and therefore f is constant. □

Proposition 24. *If n is any integer, positive or negative, and t is any real number*

$$E(nt) = E(t)^n$$

If m is a positive integer then

$$E\left(\frac{1}{m}t\right)^m = E(t)$$

and thus $E(\frac{1}{m}t)$ is the positive m-th root of $E(t)$.

Problem 20. Prove this. □

In light of Proposition 24 If r is a rational number, say $r = n/m$ with m, n integers and $m > 0$, then for a positive number a we can define

$$a^r = a^{n/m} = (a^n)^{1/m}$$

where $(a^n)^{1/m}$ is the positive m-th root of a^n. We would also like to define a^r when r is irrational. Note that when $r = m/n$ and $a = E(t)$, then Proposition 24 shows us that

$$(1) \qquad a^r = E(t)^{n/m} = (E(t)^n)^{1/m} = E(nt)^{1/m} = E\left(\frac{1}{m}nt\right) = E(rt).$$

But $E(rt)$ makes sense for all real numbers r. We now formalize all this.

Definition 25. We now officially define ***logarithm*** of a positive number x to be

$$\ln(x) = L(x) = \int_1^x \frac{dt}{t},$$

the number e to be

$$e = E(1)$$

and for any real number x we define the power e^x by

$$e^x = E(x).$$

□

Definition 26. Let $a > 0$. Then for any real number r define

$$a^r = e^{r\ln(a)}.$$

(Note if $a = E(t) = e^t$ then $\ln(a) = t$ and this becomes $a^r = e^{r\ln(a)} = e^{rt} = E(rt)$ which agrees with our preliminary definition (1).) □

Proposition 27. *If $a > 0$ and $r = n/m$ is a rational number with $m > 0$, then*

$$a^r = (a^n)^{1/m}$$

so that our definition agrees with what it should be on the rational numbers.

Problem 21. Prove this. □

Proposition 28. *With these definition the following hold*

(a) *If $a > 0$ then for all $r, s \in \mathbb{R}$*

$$a^r a^s = a^{r+s}, \quad \frac{a^r}{a^s} = a^{r-s}.$$

and,

$$(a^r)^s = a^{rs}.$$

(b) *If $r \in \mathbb{R}$ and $a, b > 0$ then*

$$a^r b^r = (ab)^r.$$

Problem 22. Prove this. □

Proposition 29. *Let r be a real number and on define $f\colon (0, \infty) \to (0, \infty)$ by*

$$f(x) = x^r.$$

Then f is differentable and

$$f'(x) = rx^{r-1}.$$

Problem 23. Prove this. *Hint:* We know that $E(x) = e^x$ is differentiable with derivative $E'(x) = E(x)$ and that $\ln(x)$ is differentiable with $\frac{d}{dx}\ln(x) = 1/x$. Thus $f(x) = e^{r\ln(x)} = E(r\ln(x))$ is a composition of differentable functions. Use the chain rule to derive the formula for $f'(x)$. □

Proposition 30. *Let a be a positive real number and define $g\colon \mathbb{R} \to (0, \infty)$ by*

$$g(x) = a^x.$$

Then g is differentable and

$$g'(x) = \ln(a)a^x.$$

Problem 24. Prove this. □

4. Some problems on Riemann sums.

Problem 25. Find the following limits by interrupting them as Riemann sums.

(a) $\displaystyle\lim_{n\to\infty} \frac{1}{n\sqrt{n}} \sum_{k=1}^{n} \sqrt{k}.$

(b) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{2n} \frac{1}{1 + (k/n)^2}.$

(c) $\displaystyle\lim_{n\to\infty} \frac{1}{n} \sum_{k=-n}^{n} \cos(k\pi/2).$