Math 555 Homework

Definition 1. Let $[a, b]$. Then a ***partition***, $\mathcal{P}$, of I is a finite sequence of points $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_{n-1} \leq x_n = b$. We will use the notation

$$I_j := [x_{j-1}, x_j]$$

is the j-th interval in the partiation and

$$\Delta x_j = (x_j - x_{j-1})$$

is the length of I_j. □

Definition 2. Let $\delta > 0$ and let $\mathcal{P}$ be a partition of $I = [a, b]$. Then the partition is δ***-fine*** iff $\Delta x_j < \delta$ for all j. We write this as

$$\mathcal{P} < \delta.$$

□

Proposition 3. *For all intervals $[a, b]$ and $\delta > 0$ there is at least one δ fine partition of $[a, b]$.*

Problem 1. Prove this. □

Definition 4. A ***partition with selection***, $\mathcal{S}$, of $[a, b]$ is an ordered pair $(\mathcal{P}, \{x_1^*, x_2^*, \ldots, x_{n-1}^*, x_n^*)\}$ with $x_j^* \in I_j$ for all j. □

Definition 5. If $\mathcal{S}$ is a partition with selection of $[a, b]$ and $f : [a, b] \to \mathbb{R}$ is a function then the ***Riemann sum*** determined by f and $\mathcal{S}$ is

$$S(f, \mathcal{S}) = \sum_{j=1}^{n} f(x_j^*) \Delta x_j.$$

□

We proved the following in class.

Proposition 6. *Let $\mathcal{S}$ be a partition with selection for $[a, b]$, $f, g : [a, b] \to \mathbb{R}$ and c a constant. Then*

$$S(c, \mathcal{S}) = c(b - a)$$
$$S(f + g, \mathcal{S}) = S(f, \mathcal{S}) + S(g, \mathcal{S})$$
$$S(cf, \mathcal{S}) = cS(f, \mathcal{S})$$

and if $f \leq g$ on $[a, b]$ the inequality

$$S(f, \mathcal{S}) \leq S(g, \mathcal{S})$$

holds. □

Definition 7. A function $f : [a, b] \to \mathbb{R}$ is ***Riemann integrable*** with integral I iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that for all partitions with selection $\mathcal{S}$

$$\mathcal{S} < \delta \qquad \Longrightarrow \qquad |S(f, \mathcal{S}) - I| < \varepsilon.$$

We have seen that the value of I is unique and we denote it by

$$I = \int_a^b f(x)\,dx. \qquad \square$$

Proposition 8 (Propostion of Julio Diaz)**.** *If $f = c$ is constant on $[a, b]$ then f is Rieman integrable on $[a, b]$ and*

$$\int_a^b c\,dx = c(b - a). \qquad \square$$

Proposition 9. *If f and g are both Riemann integrable on $[a, b]$ then so is the sum $f + g$ and*

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

Problem 2. Prove this. $\square$

Proposition 10. *If f is Riemann integrable on $[a, b]$ and c is a constant then cf is Riemann integrable on $[a, b]$ and*

$$\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx.$$

Problem 3. Prove this. $\square$

Proposition 11. *Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be distinct points in $[a, b]$ and $c_1, c_2, \ldots, c_n$ any real numbers. Let $f\colon [a, b] \to \mathbb{R}$ be the function defined by*

$$f(x) = \begin{cases} 0, & x \neq \alpha_k \text{ for any } k; \\ c_k & x = \alpha_k. \end{cases}$$

Then f is integrable and

$$\int_a^b f(x)\,dx = 0.$$

Problem 4. Prove this. $\square$

Problem 5. Let $a < \alpha < b$ and let f be the function defined on $[a, b]$ by

$$f(x) = \begin{cases} c_1, & a \leq x < \alpha; \\ c_2 & x = \alpha \leq x \leq b. \end{cases}$$

where c_1, c_2 are arbitrary constants.

(a) Graph $y = f(x)$ in the case $[a, b] = [2, 5]$, $\alpha = 3$, $c_1 = 4$, $c_2 = -3$. Based on your graph what do you think the value of $\int_2^5 f(x)\,dx$ should be? $\square$

Back to the general case.

(b) What do you think the value of $\int_a^b f(x)\,dx$ should be? *Hint:* The answer is $c_1(\alpha - a) + c_2(b - \alpha)$.

Let $\mathcal{P} = \{a = x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, $\{x_1^*, x_2^*, \ldots, , x_n^*)\}$ a selection for $\mathcal{P}$ and and $\mathcal{S} = (\mathcal{P}, \{x_1^*, x_2^*, \ldots, , x_n^*)\})$ the corresponding partition with selection.

(b) Show that if $x_j^* < \alpha$ that

$$f(x_j^*)\Delta x_j = c_1 \Delta x_j$$

and if $x_{j-1}^* > \alpha$ then

$$f(x_j^*)\Delta x_j = c_2 \Delta x_j$$

(c) If $x_{j-1} < \alpha < x_j$ show

$$\begin{aligned} S(f, \mathcal{S}) &- \Big(c_1(\alpha - a) + c_2(b - \alpha)\Big) \\ &= f(x_j^*)\Delta x_j - \Big(c_1(\alpha - x_{j-1}) + c_2(x_j - \alpha)\Big) \\ &= f(x_j^*)(x_j - x_{j-1}) - \Big(c_1(\alpha - x_{j-1}) + c_2(x_j - \alpha)\Big) \\ &= f(x_j^*)\Big((x_j - \alpha) + (\alpha - x_{j-1})\Big) - (c_1(\alpha - x_{j-1}) + c_2(x_j - \alpha)) \\ &= (f(x_j^*) - c_1)((\alpha - x_{j-1})) + (f(x_j^*) - c_2)(x_j - \alpha)) \end{aligned}$$

and therefore

$$\Big| S(f, \mathcal{S}) - \Big(c_1(\alpha - a) + c_2(b - \alpha)\Big) \Big| \leq |c_2 - c_1| \Delta x_j .$$

(You should be able to draw a picture that makes this inequality clear.)

(d) Show that f is Riemann integrable. (Note that you still have to consider the case where $x_j = \alpha$ for some j.) □

Let $a < b < c$ and let f be Riemann integrable on $[a, b]$. Extend f to $[a, c]$ to a function g on $[a, c]$ by letting g be zero on $(b, c]$. Explictly

$$g(x) = \begin{cases} f(x), & x \in [a, b]; \\ 0, & x \in (b, c]. \end{cases}$$

Let $\mathcal{S} = (\mathcal{P}, \{x_1^*, x_2^*, \ldots, x_n^*\})$ be a partition with selection of $[a, c]$, where $\mathcal{P}$ is given by $a = x_0 \leq x_1, \leq x_2 \leq \cdots \leq x_n = c$. There is a unique m with

$$x_{m-1} < b \leq x_m .$$

Define a partition $a = \tilde{x}_0 \leq \tilde{x}_1 \leq \cdots \tilde{x}_{m-1} \leq \tilde{x}_m = b$ by

$$\tilde{x}_j = x_j \quad \text{for} \quad 1 \leq j \leq m - 1 \qquad \text{and} \qquad \tilde{x}_m = b.$$

We make a selection $\{\tilde{x}_0^*, \tilde{x}_1^*, \ldots, \tilde{x}_{m-1}^*, \tilde{x}_m^*\}$ for this partition by

$$\tilde{x}_j^* = x_j^* \quad \text{for} \quad 1 \leq j \leq m - 1 \qquad \text{and} \qquad \tilde{x}_m^* = b$$

Then

$$\begin{aligned}
S(g,\mathcal{S}) &= \sum_{j=1}^{n} g(x_j^*)\Delta x_j \\
&= \sum_{j=1}^{m} g(x_j^*)\Delta x_j \qquad\qquad (\text{As } g(x_j^*)=0 \text{ for } j\ge m.) \\
&= \sum_{j=1}^{m-1} g(x_j^*)\Delta x_j + g(x_m^*)\Delta x_m \\
&= \sum_{j=1}^{m-1} f(\tilde{x}_j^*)\Delta \tilde{x}_j + f(x_m^*)\Delta x_m \\
&= \sum_{j=1}^{m-1} f(\tilde{x}_j^*)\Delta \tilde{x}_j + f(\tilde{x}_m^*)\Delta \tilde{x}_m - f(\tilde{x}_m^*)\Delta \tilde{x}_m + f(x_m^*)\Delta x_m \\
&= \sum_{j=1}^{m} f(\tilde{x}_j^*)\Delta \tilde{x}_j + \Big(f(x_m^*)\Delta x_m - f(\tilde{x}_m^*)\Delta \tilde{x}_m\Big) \\
&= S(f,\tilde{\mathcal{S}}) + \Big(f(x_m^*)\Delta x_m - f(\tilde{x}_m^*)\Delta \tilde{x}_m\Big).
\end{aligned}$$

This gives

$$\left|S(g,\mathcal{S}) - S(f,\tilde{\mathcal{S}})\right| = |f(x_m^*)\Delta x_m - f(\tilde{x}_m^*)\Delta \tilde{x}_m|$$

If f is bounded, say $|f|\le B$ and the partition $\mathcal{S}$ is δ fine, then the partition $\tilde{\mathcal{S}}$ will also be δ fine. Thus

$$\left|S(g,\mathcal{S}) - S(f,\tilde{\mathcal{S}})\right| \le |f(x_m^*)|\Delta x_m + |f(\tilde{x}_m^*)|\Delta \tilde{x}_m \le 2B\delta.$$

As f is Riemann integrable on $[a,b]$ there is a $\delta_1>0$ such that for all partitions with selection $\mathcal{S}_1$ of $[a,b]$

$$\mathcal{S}_1 < \delta \qquad \Longrightarrow \qquad \left|S(f,\mathcal{S}_1) - \int_a^b f(x)\,dx\right| < \frac{\varepsilon}{2}.$$

Let

$$\delta = \min\left\{\delta_1, \frac{\varepsilon}{4B}\right\}.$$

Then for any δ fine partition, $\mathcal{S}$, of $[a,c]$

$$\begin{aligned}
\left|S(g,\mathcal{S}) - \int_a^b f(x)\,dx\right| &= \left|S(g,\mathcal{S}) - S(f,\tilde{\mathcal{S}}) + S(f,\tilde{\mathcal{S}}) + \int_a^b f(x)\,dx\right| \\
&\le \left|S(g,\mathcal{S}) - S(f,\tilde{\mathcal{S}})\right| + \left|S(f,\tilde{\mathcal{S}}) + \int_a^b f(x)\,dx\right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

Thus g is Riemann integrable on $[a, c]$ and

$$\int_a^c g(x)\,dx = \int_a^b f(x)\,dx.$$