Math 532/736I **Test 2** Name: Answer Key

1. State the three axioms of projective geometry.

Axiom 1. For any pair of distinct points there is unique line incident with these points. ☐

Axiom 2. For any pair of distinct lines there is unique point incident with these lines. ☐

Axiom 3. There are four points no three of which are incident with the same line. ☐

2. Define the following:

(a) The points A, B, and C are ***affine independent***.

Solution: They are not colinear. ☐

(b) P is an ***affine combination*** of the points P_1, P_2, P_3, and P_4.

Solution: There are scalars α_1, α_2, α_3, and α_4 with $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = 1$ and

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_2 + \alpha_4 P_4.$$

☐

(c) The map $f\colon \mathbb{A}^2 \to \mathbb{A}^2$ is an ***affine map***.

Solution: For any points $P, Q \in \mathbb{A}^2$ and scalar t the equality

$$f((1-t)P + tQ) = (1-t)f(P) + tf(Q)$$

holds. ☐

Alternate Solution: For any points $P, Q \in \mathbb{A}^2$ and scalars α and β with $\alpha + \beta = 1$ the equality

$$f(\alpha P + \beta Q) = \alpha f(P) + \beta f(Q)$$

holds. ☐

3. Recall that the lines in $\mathbb{A}^2$ are the sets of the form

$$L(a, b, c) = \{(x, y) : ax + by + c = 0\}$$

where a and b are not both zero. Let $A = (3, 2)$ and $B = (1, 4)$, and $\ell = L(1, -1, 3)$. Find $\ell \cap \overleftrightarrow{AB}$.

Solution: The points on $\overleftrightarrow{AB}$ are affine combinations of A and B, that is they are of the form

$$(x, y) = (1-t)A + tB = (1-t)(3, 2) + t(1, 4) = (3 - 2t, 2 + 2t).$$

For one of these points to be on ℓ we have

$$x - y + 3 = 1(3 - 2t) - 1(2 + 2t) + 3 = 4 - 4t = 0$$

and therefore $t = 1$. This gives $(x, y) = (3 - 2(1), 2 + 2(1)) = (1, 4)$.

$\ell \cap \overleftrightarrow{AB} =$ $(1, 4)$

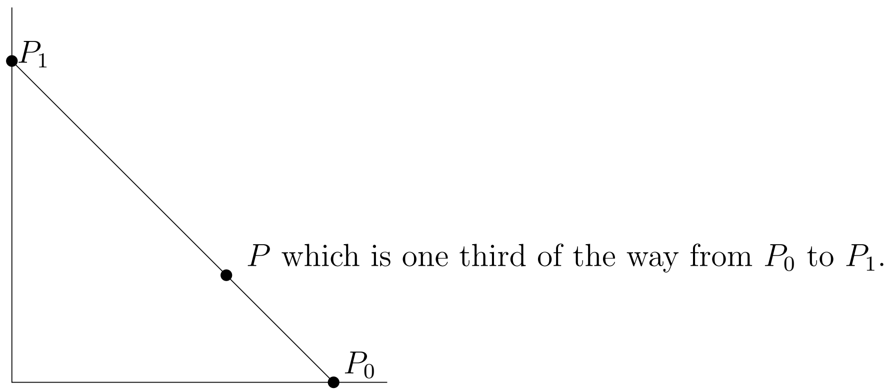
4. Definition. The point P is ***one third of the way from*** P_0 ***to*** P_1 iff

$$P = \frac{2}{3}P_0 + \frac{1}{3}P_1.$$

☐

(a) For the points $P_0 = (1, 0)$ and $P_1 = (0, 1)$ draw the point that is one third of the way from P_0 to P_1.

Solution:



□

(b) Let $f \colon \mathbb{A}^2 \to \mathbb{A}^2$ be an affine map and P the point that is one third of the way from P_0 to P_1. Prove that $f(P)$ is one third of the way from $f(P_0)$ to $f(P_1)$.

Solution: We are given that

$$P = \frac{2}{3}P_0 + \frac{1}{3}P_1$$

and that if α and β are scalars with $\alpha + \beta = 1$ then

$$f(\alpha P_0 + \beta P_1) = \alpha f(P_0) + \beta f(P_1).$$

We wish to show

$$f(P) = \frac{2}{3}f(P_0) + \frac{1}{3}(P_1).$$

This follows from the equations just given by letting $\alpha = \frac{2}{3}$ and $\beta = \frac{1}{3}$. □

5. Theorem. *If P_1, P_2, and P_3 are affinely independent points in $\mathbb{A}^2$, then any point $P \in \mathbb{A}^2$ is affine combination of P_1, P_2, and P_3.*

Use this theorem to prove that if f and g are affine maps that agree on an affinely independent set, then $f(P) = g(P)$ for all P.

Solution: Let P_0, P_1, and P_2 be three affinely independent points on which f and g agree. That is

$$f(P_1) = g(P_1), \qquad f(P_2) = g(P_2), \qquad f(P_3) = g(P_3).$$

Let P be any point of $\mathbb{A}^2$. Then we

6. (15 points) Let $f \colon \mathbb{A}^2 \to \mathbb{A}^2$ be a bijective affine map with inverse f^{-1}. Prove f^{-1} is also an affine map.

7. (15 points) (a) State ***Desargues' Theorem***.

(b) Draw a picture to illustrate Desargues' Theorem.