Take home part of Final.

This will be due at the beginning at the beginning of the period when you take the final. As with the last take home exam, write your solutions as if you the reader was someone who did not know how to do the problem and wanted to understand it. In particular this means including a English explanations of what you are doing. A problem that only has formulas and no English will lose points.

You may use your notes and other non-human sources (for example to look up trig identities). I do not mind if you talk to each other a bit about the basic ideas, but do not ask for information from people not in the class.

To begin we put the punch line on some of the recent results we have proven. In particular we have shown

Theorem 1. *Every rigid motion of $\mathbb{R}^2$ is an affine map. That is if T is a rigid motion, then*

$$T(\vec{v}) = A\vec{v} + \vec{b}$$

for some matrix A and vector $\vec{b}$. □

We now want to say more about the matrix A, in particular we want to be able to say that it is an orthogonal matrix. This will be based on the following, which we have also proven.

Theorem 2. *The A is a matrix such that $\|A\vec{v}\| = \|\vec{v}\|$ for all $\vec{v}\|$, then A is orthogonal.* □

We now use this to prove

Theorem 3. *Let $T(\vec{v}) = A\vec{v} + \vec{b}$ as be a rigid motion where where A is a matrix and $\vec{b}$ is a vector. Then A is an orthogonal matrix.*

1. (10 points) Prove this. *Hint:* Let $S = T_{-\vec{b}}$ be the translation by $-\vec{b}$. Then S is a rigid motion and therefore so is the composition $S \circ T$. Show $S \circ T(\vec{v}) = A\vec{v}$. So you can now use Theorem 2. □

Recall that the rotation of angle α about the origin given by the matrix

$$R_\alpha = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}.$$

This is the rotation in the positive (counterclockwise) direction.

2. (5 points) Show that if α is not a multiple of 2π (that is $\mathbb{R}_\alpha \neq I$ where I is the identity matrix), then

$$\det(I - R_\alpha) \neq 0.$$

Explicitly you want to show that the determinant of

$$I - R_\alpha = \begin{bmatrix} 1 - \cos\alpha & +\sin\alpha \\ -\sin\alpha & 1 - \cos\alpha \end{bmatrix}$$

is only zero when α is an integer multiple of 2π. *Hint:* You can use that fact that if $\cos\alpha = 1$, then α is an integer multiple of 2π. □

3. (10 points) Let T be the rigid motion

$$T(\vec{v}) = R_\alpha \vec{v} + \vec{a}.$$

Assume that α is not an integer multiple of 2π. Show that T has a unique fixed point. (By definition a ***fixed point*** of T is a point $\vec{v}$ such that $T(\vec{v}) = \vec{v}$). *Hint:* In Problem 1 you have shown that $\det(I - R_\alpha) \neq 0$. And we know that this implies that the inverse $(I - R_\alpha)^{-1}$ exists. Therefore if you have an equation $(I - R_\alpha)\vec{v} = \vec{b}$ for some vector $\vec{b}$, then you can solve for $\vec{v}$ to get $\vec{v} = (I - R_\alpha)^{-1}\vec{b}$. □

4. (10 points) Again let T be given by

$$T(\vec{v}) = R_\alpha \vec{v} + \vec{a}$$

and let $\vec{c}$ be the fixed point of T. Show T can be written as

$$T(\vec{v}) = \vec{c} + R_\alpha(\vec{v} - \vec{c}).$$

Hint: As $\vec{c}$ is a fixed point we have $T(\vec{c}) = \vec{c}$, that is

$$T(\vec{c}) = R_\alpha \vec{c} + \vec{a} = \vec{c}.$$

You can now solve for $\vec{a}$ and plug this back into the definition of T and rearrange a bit. □

If T is a rotation, then we call the unique fixed point of T the ***center of*** T, or say that T is ***rotation about*** $\vec{c}$.

5. (5 points) Let S and T be given by

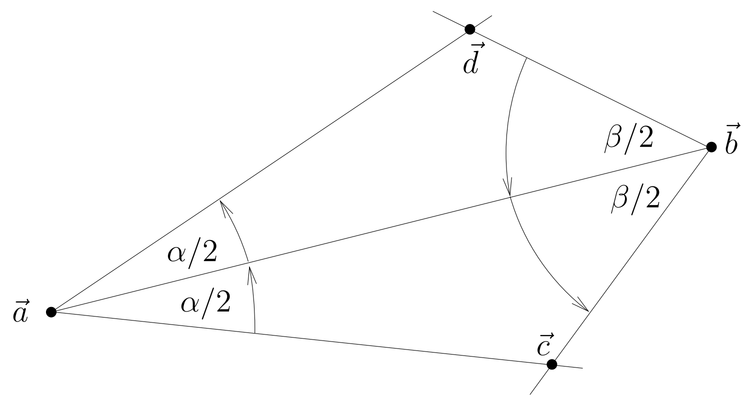
$$S(\vec{v}) = R_\alpha(\vec{v}) + \vec{a} \qquad \text{and} \qquad T(\vec{v}) = R_\beta(\vec{v}) + \vec{b}.$$

That is S is a rotation by α and T is a rotation by β. Show that the composition $S \circ T$ is a rotation by $\alpha + \beta$. (You can use that $R_\alpha R_\beta = R_{\alpha+\beta}$.) □

6. (10 points) Let S be the rotation with center $\vec{a}$ and rotation angle α and T the rotation with center $\vec{b}$ and rotation angle β. If you want formulas this means that

$$S(\vec{v}) = \vec{a} + R_\alpha(\vec{v} - \vec{a}) \qquad \text{and} \qquad T(\vec{v}) = \vec{b} + R_\beta(\vec{v} - \vec{b}).$$

We know that the composition $T \circ S$ is a rotation by $\alpha + \beta$, but we would like to also know where the center of $T \circ S$ is located. You can try to do this using formulas, but that is a mess. Here is a geometric way to see what is going on.



The arrows on the angles show the directions of the rotations. We have from the AAS criterion for congruence of triangles that $\|\vec{d}-\vec{a}\| = \|\vec{c}-\vec{a}\|$ and $\|\vec{c}-\vec{b}\| = \|\vec{d}-\vec{b}\|$.

(a) Explain why $S(\vec{c}) = \vec{d}$.

(b) Explain why $T(\vec{d}) = \vec{c}$.

(c) Now combine (a) and (b) to show $T \circ S(\vec{c}) = \vec{c}$ and therefore $\vec{c}$ is the center of $T \circ S$.