Modern Geometry Homework.

1. One to one correspondences and the cardinality of sets.

Definition 1. Let $f\colon A \to B$ be a function between sets.

(a) f is ***one-to-one*** (or ***injective***) iff for all $a_1, a_2 \in A$ with $a_1 \neq a_2$ we have $f(a_1) \neq fa_2)$. (Or what is the same thing if $a_1, a_2 \in A$ and $f(a_1) = f(a_2)$ then $a_1 = a_2$.)

(b) f is ***onto*** (or s***urjective***) iff for all $b \in B$ there is an $a \in A$ with $f(a) = b$.

(c) f is ***one-to-one and onto*** (or ***bijective***, or a ***one-to-one correspondence***) iff f both one-to-one and onto. $\square$

Definition 2. If $f\colon A \to B$ and $g\colon B \to C$, then the ***composition*** of g and f, denoted by $g \circ f$ is the function $g \circ f\colon A \to C$ given by

$$g \circ f(x) := g(f(x)).$$

$\square$

Proposition 3. *Let $f\colon A \to B$ and $g\colon B \to C$ be functions. Then*

(a) If f and g are both one-to-one, then so is $g \circ g$.

(b) If f and g are both onto, then so is $g \circ f$.

(c) if f and g are both bijective, then so is $g \circ f$.

Problem 1. Prove this. *Hint:* Note that once you have done parts (a) and (b) you get part (c) for free. For parts (a) and (b) it is mostly a matter of writing down the definitions and following your nose. For example in (b) if $c \in C$, as g is onto, there is a $b \in B$ with $g(b) = c$. But then as f is onto there is an $a \in A$ with $f(a) = b$. Then what is $(g \circ f)(a) = g(f(a))$? $\square$

Definition 4. Let A and B be sets. Then A and B ***have the same number*** of elements (or have the same ***cardinality***) iff there is a one-to-one correspondence $f\colon A \to B$. $\square$

Proposition 5. *If A and B have the same cardinality and B and C have the same cardinality, then A and C have the same cardinality.*

Problem 2. Prove this. *Hint:* Proposition 3 part (c). $\square$

Proposition 6. *Let $f\colon A \to B$ and $g\colon B \to A$ be functions. Assume*

$$g(f(a)) = a \quad \text{for all } a \in A, \qquad f(g(b)) = b \quad \text{for all } b \in B.$$

*Then both f and g are one-to-one and onto. In this case we say that f is the **inverse** of g (and that g is the inverse of f).*

Problem 3. Prove this. *Hint:* As the hypothesis are symmetric in f and g it is enough to prove that f is bijective for, by reversing the roles of f and g, the same proof would show that g is bijective.

To show that f is injective assume that $f(a_1) = f(a_2)$. Then $g(f(a_1)) = g(f(a_2))$. But what are $g(f(a_1))$ and $g(f(a_1))$?

To show that f is onto, let $b \in B$. Then what is $f(g(b))$?

In this class many, if not most, of the one-one correspondences we well construct are based on drawing a line trough a point on one set and seeing where it interests anther set. We will use the notation

$$\overleftrightarrow{PQ} = \text{the line through the points } P \text{ and } Q.$$

and

$$\overrightarrow{PQ} = \text{the ray starting at } P \text{ and passing through } Q.$$

Here are a couple of examples of this idea.

Problem 4. Let O be the origin in the plane $\mathbb{R}^2$

(a) Draw the circle, C, defined by $x^2 + y^2 = 1$ and the ellipse, E, defined by $\frac{x^2}{4} + \frac{y^2}{9} = 1$ on the same axis.

(b) Let $P, P' \neq O$ be points such that P' is on the ray $\overrightarrow{OP}$. Draw a picture which shows why $\overrightarrow{OP} = \overrightarrow{OP'}$.

(c) Define a function $f\colon C \to E$ be letting $f(P)$ by the point on the ray $\overrightarrow{OP}$ intersects E. That is

$$f(P) = E \cap \overrightarrow{OP}.$$

Note by part (b) we have $\overrightarrow{OP} = \overrightarrow{Of(P)}$. Choose three points P_1, P_2 and P_3 on C and draw their images under f on E.

(d) Define a function $g\colon E \to C$ be letting $g(Q)$ by the point on the ray $\overrightarrow{OQ}$ intersects C, that is

$$g(Q) = C \cap \overrightarrow{OQ}.$$

Again we can use part(b) to see that $\overrightarrow{OQ} = \overrightarrow{Og(Q)}$ Choose three points Q_1, Q_2 and Q_3 on E and draw their images under g.

(e) Explain why $g(f(P)) = P$ for all $P \in C$ and $f(g(Q)) = Q$ for all $Q \in E$. You can, and probably should, do a "proof by picture on this". But here is a more formal version of showing $f(g(P)) = P$.

$$\begin{aligned} f(g(Q)) &= E \cap \overrightarrow{Og(Q)} && \text{(Definition of } f) \\ &= E \cap \overrightarrow{OQ} && (\text{As } \overrightarrow{OQ} = \overrightarrow{Og(Q)}) \\ &= Q && \left(\begin{array}{l}\text{As } Q \text{ is on } E \text{ so it is the point} \\ \text{of intersection of } \overrightarrow{OQ} \text{ and } E\end{array}\right) \end{aligned}$$

Do a similar calculation to show $g(f(P)) = P$.

(f) Conclude that f and g are bijections and therefore C and E have the same cardinality. □

Problem 5. Figure 1 shows the circle $x^2 + y^2 = 1$ where $N = (0, 1)$ (the "North Pole"). Let S be this circle with N removed (shown in blue). Let ℓ be the x-axis (shown in red). Note for any downward pointing ray starting from N (three of these are shown in green) the ray will intersect both S and L in exactly one point.

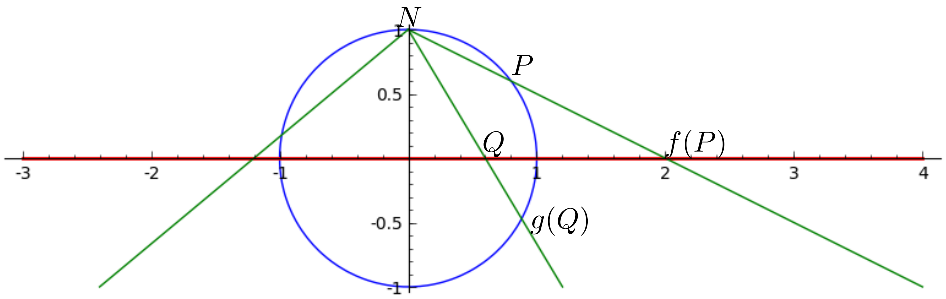

FIGURE 1

(a) Let $P, P' \neq N$ be points such that P' is on the ray $\overrightarrow{NP}$. Draw a picture which shows $\overrightarrow{NP} = \overrightarrow{NP'}$.

(b) If P is a point on S, then then the figure shows a corresponding point $f(P)$ on L. Give a verbal description of this function. In terms of a formula it is given by

$$f(P) = \ell \cap \overrightarrow{NP}.$$

Note that by part (a) we have $\overrightarrow{NP} = \overrightarrow{Nf(P)}$.

(c) If Q is a point on L, then then the figure shows a corresponding point $g(Q)$ on S. Give a verbal description. This time the formula is

$$g(Q) = S \cap \overrightarrow{NQ}.$$

And by part (a) $\overrightarrow{NQ} = \overrightarrow{Ng(Q)}$.

(d) Show these functions are inverse to each other. Again "proof by picture" if accompanied by the correct English is fine, but we can be a bit more formal. Justify the following steps.

$$\begin{aligned} g(f(P)) &= S \cap \overrightarrow{Nf(P)} \\ &= S \cap \overrightarrow{NP} \\ &= P \qquad\qquad\qquad (\text{as } P \text{ is on } S \text{ and } \overrightarrow{NP}). \end{aligned}$$

Do a similar calculation to show $f(g(Q)) = Q$. □

2. Axioms of Affine Geometry

Affine geometry is the part of Euclidean geometry that is left over when all mention of distance and angles is removed. What is left are the ideas of incidence and parallelism. A surprise is that there are more examples than just the usual plane.

We have some undefined notions. The first is ***point*** which you can think just as usual as a point in the plane. The collection of all points will be denote by $\mathbb{A}^2$. The next undefined notation is ***line***. These are special subsets of $\mathbb{A}^2$ and again think of these as lines in the usual sense. Finally there is the notion of ***incidence***. This is a relation between points and lines. To say that the point P is incidence with the line ℓ is just to say that the point P is ***on the line*** L. Other ways to say that the point P and the line ℓ are incidence are to say that ℓ ***goes through*** P, or that ℓ ***is on*** P.

The first axiom is just the familiar fact that two points determine a line.

Affine Geometry: Axiom 1. *If P and Q are distinct points of $\mathbb{A}^2$ (that is $P \neq Q$) then there is a unique line that passes through both of them.* $\square$

When we say that this line is unique we mean that there is exactly one of them. If P and Q are points of $\mathbb{A}^2$ with $P \neq Q$ then we will use the notation

$$\overleftrightarrow{PQ} := \text{The line through } P \text{ and } Q.$$

Proposition 7. *If ℓ and m are distinct lines of $\mathbb{A}^2$ then there is at most one point that is incident with both ℓ and m. (Or a little more informally, two distinct lines meet in at most one point.)*

Thus if ℓ and m are distinct lines in $\mathbb{A}^2$ either they have no point incident with both of them, or there is exactly one point that is incident with both of them. Anther restatement would be that two distinct lines can not meet in two or more points. If ℓ and m are lines we will use the notation

$$\ell \cap m = \text{Point of intersection of } \ell \text{ and } m.$$

Problem 6. Prove the last Proposition. *Hint:* Towards contradiction assume that the distinct lines ℓ and m meet in two or more points and let P and Q be two of these points. Then by first axiom of affine geometry there is exactly one line through P and Q. If you think about it a bit this gives a contradiction. $\square$

In Euclidian geometry we define two lines to be parallel if they have no point in common. But in some circumstances it is convenient to think of a line as being parallel to itself. Different text books have different conventions about this. I am going to assume that a line is parallel to itself.

Definition 8. Two lines, ℓ and m of $\mathbb{A}^2$ are ***parallel*** iff either $\ell = m$ or ℓ and m have no points in common. $\square$

Affine Geometry: Axiom 2 (Parallel Axiom)**.** *If ℓ is a line of $\mathbb{A}^2$ and P is a point that is not on ℓ, then there is a unique line m that goes through P and is parallel to ℓ. (See Figure 2.)* $\square$

Based on our definition of parallel lines we can give a slightly more general version of this.

Proposition 9. *If ℓ is any line of $\mathbb{A}^2$ and P any point, then there is a unique line m that goes through P and is parallel to ℓ.*

Problem 7. Prove this. *Hint:* The only way this differs from the Parallel Axiom is that we allow the point P to be on the line ℓ. If P is on ℓ then you just need to find a line, m, through P parallel to ℓ. $\square$

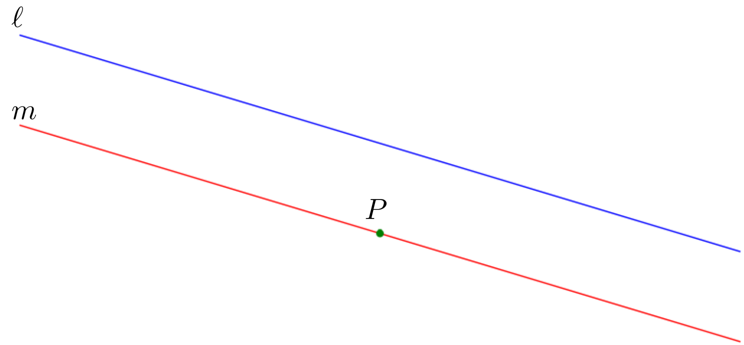

FIGURE 2. The Parallel Axiom tells us that given a line ℓ (here in blue) and a point not on ℓ (here in green) there is exactly one line m (here in red) through P that does not meet ℓ.

Proposition 10. *If ℓ_1 and ℓ_2 are parallel lines and the line $m \neq \ell_1$ intersects ℓ_1, then it also intersects ℓ_2.*

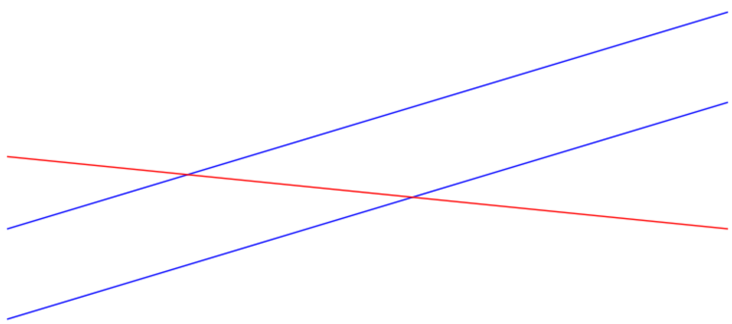

FIGURE 3. Proposition 10 tells us that if we have two parallel lines (in blue) then a third line (in red) meets one of these lines if and only if it meets the other one.

Problem 8. Prove this. *Hint:* Let P be the point where ℓ_1 and m meet. If m does not intersect ℓ_2, then how many parallels does ℓ_2 have that pass through P? Draw a picture. $\square$

Problem 9. This is a special case of a slightly more general result we will prove later. Let ℓ_1, ℓ_2 and ℓ_3 be distinct lines. Assume that ℓ_1 is parallel to ℓ_2 and that ℓ_2 is parallel to ℓ_3. Show that ℓ_1 is parallel to ℓ_3. $\square$

The last axiom of affine geometry just ensures that we have more than one line.

Affine Geometry: Axiom 3. *There exists a set of four points in $\mathbb{A}^2$ such that no three of which are all on the same line.* $\square$

We can give our first non-obvious example of an affine plane. In Figure 4 we have the four points A, B, C, D and the six lines a, b, c, d, e, f. We only count the intersections at the labeled and shaded points (this the point at the center where you might think that a and d intersect does not count). Note that in this figure any two of the points determine a unique line. For any of the six lines and a point not on the line there is a unique parallel to the line through the point. For example the parallel to d through the points A is the line a. And no three of the points are all on the same line. Therefore this gives us an example of an affine geometry that only has four points and six lines. This is the smallest possible affine geometry.

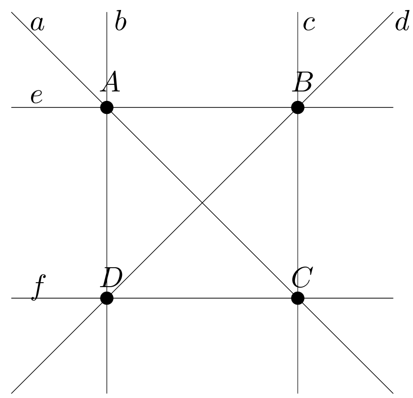

FIGURE 4. This shows four points and the six lines they determine. We only count the lines as intersecting if they both pass through one of shaded circles.

What determines the geometry of this plane is the sets of points, $\{A, B, C, D\}$ and lines $\{a, b, c, d, e, f\}$ and the incidence relations between them. In this case these are

$$\overleftrightarrow{AB} = e, \quad \overleftrightarrow{AC} = a, \quad \overleftrightarrow{AD} = b, \quad \overleftrightarrow{BC} = c, \quad \overleftrightarrow{BD} = d, \quad \overleftrightarrow{CD} = b, \tag{1}$$

and

$$\begin{aligned} A &= a \cap b = a \cap e = b \cap e, & B &= c \cap d = c \cap e = d \cap e, \\ C &= a \cap c = a \cap f = c \cap f, & D &= b \cap d = b \cap f = d \cap f. \end{aligned} \tag{2}$$

These are what defines the geometry, not the particular picture that we draw of it. Figure 5 gives anther picture of the same geometry.

Problem 10. Use Figure 4 or Figure 5 to find the following:

(a) The line through C parallel to e.
(b) The line through C parallel to d.

Lemma 11. *Let ℓ_1 and ℓ_2 be distinct lines of $\mathbb{A}^2$. Assume that a third line m (not equal to either ℓ_1 or ℓ_2 intersects both ℓ_1 and ℓ_2). Then any line n parallel to m intersects each of ℓ_1 and ℓ_2 in exactly one point. (See figure 6.)*

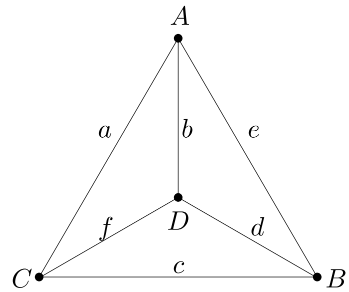

FIGURE 5. This is anther model of the affine plane of Figue 4. The reason this is the same geometry is that the same set of incidence relations (1) and (2) hold.

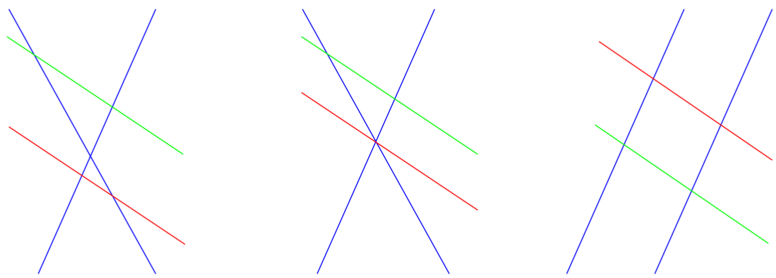

FIGURE 6. Some possible configurations of the lines in Lemma 11. Given two distinct lines (in blue) and a third line that intersects them both (in red) then any line parallel to the red line (such as the green line) also intersects each of the blue lines.

Problem 11. Prove the last Proposition. *Hint:* Towards a contradiction assume that n does not intersect ℓ_1. Let P be the point of intersection of ℓ_1 and m. Then ℓ_1 and m are parallel to n and both go through P. □

Theorem 12. *Any two lines of $\mathbb{A}^2$ have the same number of points. More precisely, for any two lines in $\mathbb{A}^2$ there is a one-to-one correspondence between their points.*

Problem 12. Prove this along the following lines. Let ℓ_1 and ℓ_2 be lines of $\mathbb{A}^2$. We wish to find a bijection between these two sets. If $\ell_1 = \ell_2$ then just use the identity map. So assume that $\ell_1 \neq \ell_2$.

(a) Show that there is a line m distinct from ℓ_1 and ℓ_2 that intersects both ℓ_1 and ℓ_2. *Hint:* As $\ell_1 \neq \ell_2$, there is a point of ℓ_1 that is not on ℓ_2 and a point of ℓ_2 that is not on ℓ_1. Let m be line through these two points.

(b) For each point $P \in \mathbb{A}^2$ let $n(P)$ be the line of $\mathbb{A}^2$ that passes through P and is parallel to m. Use Proposition 11 to show that each $n(P)$ intersects both ℓ_1 and ℓ_2. Draw a picture to illustrate this.

(c) Define a function f that takes points on ℓ_1 to points on ℓ_2 as follows and a function g that takes points on ℓ_2 to points on ℓ_1 as follows. (See Figure 7.)

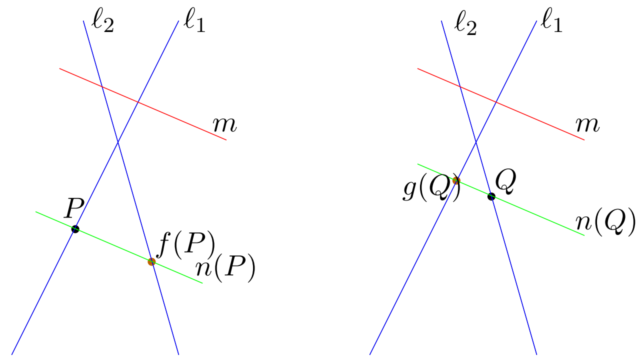

FIGURE 7. The construction of f is illustrated on the left. Given an input point P on ℓ_1, draw the line $n(P)$ through P and parallel to m. The output, $f(P)$, is the point where $n(P)$ intersects ℓ_2.

The construction of g is illustrated on the right. Given an input point Q on ℓ_2, draw the line $n(Q)$ through Q and parallel to m. Then $g(Q)$ is the point where $n(Q)$ intersects ℓ_1.

(i) If P is a point on ℓ_1, then $f(P)$ is the point where $n(P)$ intersects ℓ_2. That is $f(P) = n(P) \cap \ell_2$.

(ii) If Q is a point on ℓ_2, then $g(Q)$ is the point where $n(Q)$ intersects ℓ_1. That is $g(Q) = n(Q) \cap \ell_1$

Explain why f and g are inverse to each other. *Hint:* It is enough to show that $f(g(Q)) = Q$ and $g(f(P)) = P$. Here is one of the two calculations

$$
\begin{aligned}
f(g(Q)) &= n(g(Q)) \cap \ell_2 && \text{(definition of } f\text{)} \\
&= n(Q) \cap \ell_2 && \left(\begin{array}{l}\text{as } n(Q) = n(g(Q) \text{ because} \\ n(Q) \text{ and } n(g(Q)) \text{ are parallel} \\ \text{to } m \text{ and contain } Q \text{ and there} \\ \text{is only one such parallel.}\end{array}\right) \\
&= Q && \text{(as } Q \text{ is on both } n(Q) \text{ and } \ell_1.\text{)}
\end{aligned}
$$

Do a similar calculation so find show $g(f(P)) = P$.

(d) Use Proposition 6 to finish the proof. □

Proposition 13. *Each line of $\mathbb{A}^2$ has at least two points on it.*

Problem 13. Prove this. *Hint:* By Theorem 12 all lines have the same number of points on them. So it is enough to show that there is at least one line with two or more points on it. You should be able to do this using Affine Axiom 3. □

Theorem 14. *If some line of $\mathbb{A}^2$ has a finite number of points on it, say n, then*

(a) Every line has exactly n points on it.
(b) Each point has exactly $n+1$ lines passing through it.
(c) Each line has exactly n lines parallel to it (counting itself).
(d) $\mathbb{A}^2$ consists of exactly n^2 points.

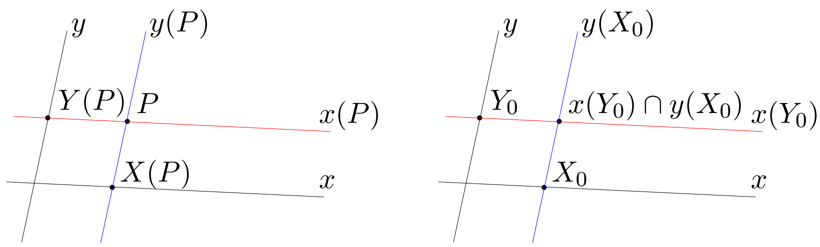

FIGURE 8. Start by choosing two intersecting lines x and y. Then for any point P of the plane let $x(P)$ be the line through P that is parallel to x and let $y(P)$ be the line through P parallel to y. Then let $X(P) = x \cap y(P)$ be the point where $y(P)$ intersects x and $Y(P) = y \cap x(P)$ be the point where $x(P)$ intersects y. This is illustrated in the figure on the left.

Conversely given points X_0 on x and Y_0 on y by looking at the point of intersection of the line through X_0 and parallel to y and the line through Y_0 and parallel to x show there is a point $P_0 = x(Y_0) \cap y(X_0)$ with $X(P_0) = X_0$ and $Y(P_0) = Y_0$. This is illustrated on the right.

(e) There are exactly $n^2 + n$ lines in $\mathbb{A}^2$.

Problem 14. Prove this. *Hints:*

(a) This follows directly from Theorem 12.

(b) Let P be a point of $\mathbb{A}^2$ and ℓ a line that does not go through P. Let m be the unique parallel to ℓ through P. For any point Q of ℓ let $\overleftrightarrow{PQ}$ be the line through P and Q. Draw a picture and explain why

$$\{m\} \cup \{\overleftrightarrow{PQ} : Q \in \ell\}$$

gives all the lines through P and why this shows that there are exactly $n+1$ of them.

(c) Let ℓ be a line of $\mathbb{A}^2$ and m a line that intersects ℓ in exactly one point. Then any line n parallel to ℓ will intersect m in a point. And given any point on m there is a line through this point parallel to ℓ. These facts can be put together to find a bijection between the lines parallel to ℓ and the points of m. Draw a picture.

(d) You should be able to use the idea suggested by Figure 8 to find a bijection between the set of points $P \in \mathbb{A}^2$ and the set of ordered pairs (X, Y) with $X \in x$ and $Y \in y$. (This should remind you of the construction of x-y coordinates in the plane.)

(e) Let ℓ be a line of $\mathbb{A}^2$. Then any point ℓ has $n+1$ lines through it by part (b). If we throw out ℓ itself, this gives that through each point of ℓ there are n lines that are not parallel to ℓ. Explain why this shows there are exactly n^2 lines of $\mathbb{A}^2$ that are not parallel to ℓ. This leaves the lines that are parallel to ℓ and by part (c) there are n such lines. $\square$

For anther example of an affine plane see Figure 9. In this example there are nine points, each line contains three points (so the n of Theorem 14 is $n = 3$) and there are $n^2 + n = 12$ lines.

In light of the couple of examples we have given so far, it is natural to ask for what number n is there an affine geometry where each line has n points.

Definition 15. If $\mathbb{A}^2$ is a an affine plane with only a finite number of points, then its ***order*** is the number of points on a line. $\square$

Thus Theorem 14 tells us that an affine plane of order n has n^2 points and $n^2 + n$ lines. For a positive result we have

Theorem 16. *If $n = p^k$ where p is a prime number and k is a positive integer, then there is an affine plane of order n.* $\square$

We will outline a proof of the above later in the term.

But not every n can be the order of a finite plane. The most general known restriction is

Theorem 17 (Bruck-Ryser theorem)**.** *If an affine plane of order n exists and n is of the form $4k+1$ or $4k+2$, then n is the sum of two squares. (That is $n = x^2 + y^2$ for some integers x and y.)* $\square$

For example 6 is of the form $4k+2$ (where $k = 1$) but 6 can not be written as $6 = x^2 + y^2$ (as you can see by a bit of trial trail and error). Thus there is no affine plane of order 6. Likewise $14 = 4(3) + 2$ is of the form $4k + 2$, and you can use trail an error to find that 14 can not be written as $14 = x^2 + y^2$ (if x or y is ≥ 4 then $x^2 + y^2 \geq 4^2 = 16$ is too large. So you only have to check the cases where $x, y = 0, 1, 2, 3$.) Therefore there is no affine plane of order 14.

Problem 15. Use the Bruck-Ryser Theorem to show that there is no affine plane of order 22. $\square$

The smallest number not covered by the last two theorems is 10. It is not a prime power, but it is the sum of two squares: $10 = 1^2 + 3^2$. It was a big deal in the 1980's, when C. W. H. Lam, L. H. Thiel, and S. Swiercz

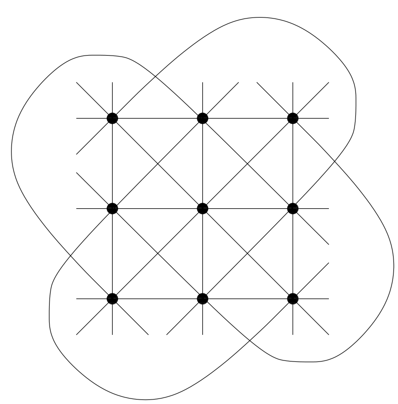

FIGURE 9. An affine plane with 9 points and 12 lines and each line containing 3 points.

[2] proved that there is no plane of order 10. Even the New York Times had an article about the result (Dec. 20, 1988, Section 3, p. 1). If you want more information Lam has a article [1] describing the history of the result. A down side to this proof is that it is not directly human readable in that it relies on a large computer search to check a vast number of special cases.

The next number not covered by the Bruck-Ryser Theorem is 12 which in not of either of the forms $4k + 1$ or $4k + 2$. If is still unknown if there is a affine plane of order 12 and this seems be be a vary hard question.

References

1. C. W. H. Lam, *The search for a finite projective plane of order* 10, Amer. Math. Monthly **98** (1991), no. 4, 305–318. MR 1103185 (92b:51013)
2. C. W. H. Lam, L. Thiel, and S. Swiercz, *The nonexistence of finite projective planes of order* 10, Canad. J. Math. **41** (1989), no. 6, 1117–1123. MR 1018454 (90j:51008)