Modern Geometry Homework.

1. Projective planes.

In planar projective geometry we have three undefined notations. The first a ***point***, the second it a ***line***, and the third is relation ***incidence*** between a point and a line. There are three axioms:

Axiom 1 (First Axiom of Projective Geometry)**.** *For any pair of distinct points there is unique line incident with these points.* □

Axiom 2 (Second Axiom of Projective Geometry)**.** *For any pair of distinct lines there is unique point incident with these lines.* □

Axiom 3 (Thrid Axiom of Projective Geometry)**.** *There are four points no three of which are incident with the same line.* □

We will often use the symbol $\mathbb{P}^2$ to denote a projective plane.

Proposition 1. *Let ℓ be a line in $\mathbb{P}^2$ and P a point of $\mathbb{P}^2$ that is not on ℓ. Let (P) be the set of all lines that are incident with P. Then there is a bijection between ℓ and $\mathcal{L}(P)$.*

Problem 1. Prove this. *Hint:* Define $f\colon \ell \to \mathcal{L}(P)$ by

$$f(Q) = \overleftrightarrow{QP}.$$

Define $g\colon \mathcal{L}(P) \to \ell$ by

$$g(m) = m \cap \ell.$$

Now

$$\begin{aligned} f(g(m)) &= \overleftrightarrow{g(m)P} && \text{(Definition of } f) \\ &= m && (m \text{ is a line incident with } P \text{ and } g(m)) \end{aligned}$$

and

$$\begin{aligned} g(f(Q)) &= m \cap f(Q) && \text{(Definition of } g) \\ &= Q && (Q \text{ is a point incident with } \ell \text{ and } f(Q)). \end{aligned}$$

This shows that f and g are bijections. Draw a picture that illustrates this proof. □

Proposition 2. *Given two distinct lines ℓ and m in the projective plane $\mathbb{P}^2$, there is a point that is not incident with either ℓ or m.*

Proof. Towards a contradiction If this is If this is false, then every point of $\mathbb{P}^2$ is on either ℓ or m (or both). By the Thrid Axiom of Projective Geometry, there are four points P_1, P_2, P_3, P_4 such that no three of them are on the same line. So we much have two of them, say P_1 and P_2, on ℓ and the other two, P_3 and P_4, on m and none of these are equal to $\ell \cap m$. Then the point $\overleftrightarrow{P_1P_3} \cap \overleftrightarrow{P_2P_4}$ is a point not on either ℓ or m, contradicting that all the points of $\mathbb{P}^2$ are on the either ℓ or m. (See Figure 1) □

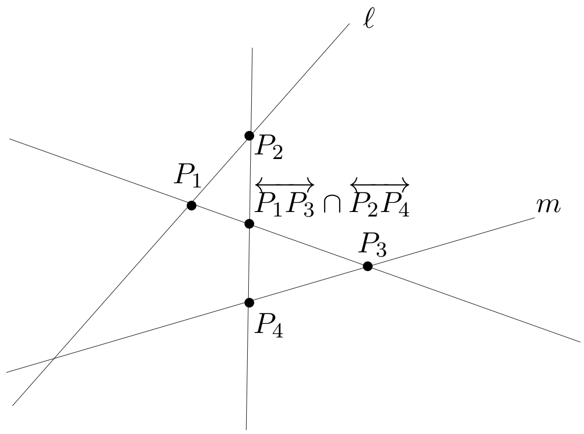

FIGURE 1

Proposition 3. *Let ℓ and m be lines in $\mathbb{P}^2$. Show that there is a bijection between ℓ and m.*

Problem 2. Prove this along the following line. If $\ell = m$, then this is clear. So assume that $\ell \neq m$ and use Proposition 1 to find a point P that is not on either ℓ or m. Then by Proposition 1 there is a bijection $f_1 \colon \ell \mathcal{L}(P)$ and a bijection $f_2 \colon m \colon \mathcal{L}(P)$. Use f_1 and f_2 to get a bijection between ℓ and m. □

Problem 3. Give anther proof of Proposition 3 along the following lines. Again we can assume that $\ell \neq m$ and that we have a point P that is on neither ℓ or m. Define a map $f \colon \ell \to m$ by

$$f(Q) = \overleftrightarrow{PQ} \cap m.$$

Draw a picture to show what f does and show that f is a bijection. □

Proposition 4. *Let P and Q be points of $\mathbb{P}^2$ and let $\mathcal{L}(P)$ be the collection of all lines that are incident with P and $\mathcal{L}(Q)$ the set of all lines incident with Q. Then there is a bijection between $\mathcal{L}(P)$ and $\mathcal{L}(Q)$.*

Problem 4. Prove this. *Hint:* One way would be to use Proposition 1. □

The following is the projective version of Theorem 14 on Homework 1.

Theorem 5. *Assume that some line ℓ of $\mathbb{P}^2$ only has a finite number of points, say $n + 1$. Then*

(a) Every line has exactly $n + 1$ points incident with it.
(b) Every point has exactly $n + 1$ lines incident with it.
(c) $\mathbb{P}^2$ contains exactly $n^2 + n + 1$ points.
(d) $\mathbb{P}^2$ contains exactly $n^2 + n + 1$ lines.

Problem 5. Prove this. □

2. Constructing projective planes from affine planes.

We recall the axioms of affine geometry.

Axiom 1 (First Axiom of Affine Geometry)**.** *Given an two distinct points, P and Q, there is a unique line, l, that is incident with both of them.*

Axiom 2 (Second Axiom of Affine Geometry)**.** *Given a line ℓ and a point P not on the line, then there is a unique line m incident with P that is parallel to ℓ.*

Axiom 3 (Thrid Axiom of Affine Geometric)**.** *There exist four points such that no three of them are on the same line.*

Let $\mathbb{A}^2$ be an affine plane. We will extend this to a projective plane $\mathbb{P}^2$. For each line ℓ we add a point P_ℓ to ℓ and call the point the ***point at infinity*** on ℓ, or the ***ideal point*** on ℓ. Given two line ℓ and m we say they have the same point at infinity exactly when they are parallel. That is

$$P_\ell = P_m \qquad \Longleftrightarrow \qquad \ell \parallel m.$$

For each line ℓ of $\mathbb{A}^2$ let ℓ^* be ℓ with P_ℓ attached.

We add one line, ℓ_∞, the line at infinity.

The incidence relations are that if P is a