Modern Geometry Homework.

1. Rigid motions of the line.

Let $\mathbb{R}$ be the real numbers. We define the ***distance*** between $x, y \in \mathbb{R}$ by

$$\text{distance between } x \text{ and } y = |x - y|$$

where

$$|z| = \begin{cases} z, & z \geq 0; \\ -z, & z < 0. \end{cases}$$

is the usual absolute value.

Proposition 1. *If $a, b \in \mathbb{R}$ are distinct points and x is an equal distance from a and b, that is*

$$|x - a| = |x - b|$$

then x is the midpoint between a and b. That is

$$x = \frac{a + b}{2}.$$

Problem 1. Prove this. *Hint:* One way is to square both sides of the equation $|x - a| = |x - b|$ to get rid of the absolute values, that is

$$(x - a)^2 = (x - b)^2$$

and you can now solve this for x. □

Corollary 2. *If*

$$|x - 1| = |x + 1|$$

then $x = 0$.

Proof. This is the special case of Proposition 1 with $a = -1$ and $b = 1$. □

Definition 3. A rigid motion of $\mathbb{R}$ is a function $T\colon \mathbb{R} \to \mathbb{R}$ that preserves distances. That is

$$|T(x) - T(y)| = |x - y|$$

for all $x, y \in \mathbb{R}$. □

We fist give some elementary properties of rigid motions that follow directly from the definition.

Proposition 4. *Let S and T be rigid motions. Then the composition $S \circ T$ is also a rigid motion. (That is $|S(T(x)) - S(T(y))| = |x - y|$ for all $x, y \in \mathbb{R}$.)*

Problem 2. Prove this. □

Proposition 5. *If $T\colon \mathbb{R} \to \mathbb{R}$ is a rigid motion that has an inverse T^{-1}, then the inverse T^{-1} is also a rigid motion.*

Problem 3. Prove this. □

We now give some examples.

Definition 6. Let $a \in \mathbb{R}$. Then the ***translation*** by a is the map

$$T_a(x) = x + a.$$

$\square$

Proposition 7. *Each translation is a rigid motion. Moreover if $T = T_a$ is a translation, it has the property it moves each point by the same amount. That is there is a constant c such that*

$$|T(x) - x| = c$$

Problem 4. Prove this. $\square$

Definition 8. Let $a \in \mathbb{R}$. Then the ***reflection about*** a is the map

$$R_a(x) = 2a - x.$$

$\square$

Proposition 9. *Each reflection is a rigid motion. The reflection R_a has the property that*

$$R_a(a) = a$$

and for any x we have

$$|R_a(x) - a| = |x - a|.$$

Problem 5. Prove this and draw some pictures illustrating the property $|R_a(x) - a| = |x - a|$. $\square$

Proposition 10. *The translation T_a has T_{-a} as in inverse. The reflection R_a is its own inverse.*

Problem 6. Prove this. $\square$

Proposition 11. *The following hold:*

(a) The composition of two translations is a translation.
(b) The composition of a translation with a reflection is a reflection.
(c) The composition of a reflection and translation is a reflection.
(d) The composition of two reflections is a translation.

Problem 7. Prove this by finding formulas for all of the following:

(a) $T_a \circ T_b$,
(b) $T_a \circ R_b$,
(c) $R_b \circ T_a$,
(d) $R_a \circ R_b$. $\square$

We now will show that we have found all the rigid motions.

Proposition 12. *Let S be a rigid motion that has the farther property that*

$$S(0) = 0.$$

Then either

$$S(x) = x \qquad \text{(for all } x\text{, that is } S = T_0 \text{ is the translation by } 0\text{)}$$

or

$$S(x) = -x \qquad \textit{(for all } x\textit{, that is } S = R_0 \textit{ is the reflection in } 0.\textit{)}$$

Problem 8. Prove this along the following lines.

(a) Show that for all x

$$S(x) = \pm x$$

where the choice of $+$ or $-$ may depend on x. *Hint:* As $S(0) = 0$ you can use the the defining property of being a rigid motion to conclude that $|S(x)| = |S(x) - S(0)| = |x - 0| = |x|$.

(b) Either $S(1) = 1$ or $S(x) = -1$.

(c) If $S(1) = 1$, then $S(x) = x$ for all x, that is $S = T_0$. Here is the proof we gave in class, which you do not have to turn in. Assume towards a contradiction, that there is some $x \neq 0$ such that $S(x) \neq x$. Then by part (a) $S(x) = -x$. Then

$$|S(x) - S(1)| = |x - 1|$$

can be combined with

$$|S(x) - S(1)| = |-x - 1| = |x + 1|$$

which in turn can be combined with Corollary 2 to conclude that $x = 0$, contradicting our assumption that $x \neq 0$.

(d) If $S(1) = -1$, then $S(x) = -x$ for all x. That is $S = R_0$, the reflection in the origin. *Hint:* This can be done in same way as part (c).

(e) Put the pieces together to complete the proof. □

Theorem 13. *If T is a rigid motion of $\mathbb{R}$, then T is either a translation or a reflection.*

Problem 9. Prove this. *Hint:* Let $S\colon \mathbb{R} \to \mathbb{R}$ be the map

$$S(x) = T(x) - T(0).$$

(a) Show that S is a rigid motion with $S(0) = 0$.

(b) Show that either $S(x) = x$ for all x, that is $S = T_0$, or $S(x) = -x$ for all x, that is $S = R_0$.

(c) Use that $T(x) = S(x) + T(0)$ to finish the proof. □

And here is a problem to relate this to what we have done before.

Problem 10. Show that all translations and reflection are affine maps. □

2. Rigid motions of the plane.

We first recall some vector algebra. Let $\vec{a} = (a_1, a_2)$ and $\vec{b} = (b_1, b_2)$ be vectors in $\mathbb{R}^2$. Then the ***inner product*** of $\vec{a}$ and $\vec{b}$ is

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2.$$

and we define the ***length*** a vector $\vec{a}$ to be

$$\|\vec{a}\| = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{a_1^2 + a_2^2}.$$

This is the natural definition based on the Pythagorean Theorem as the following figure shows:

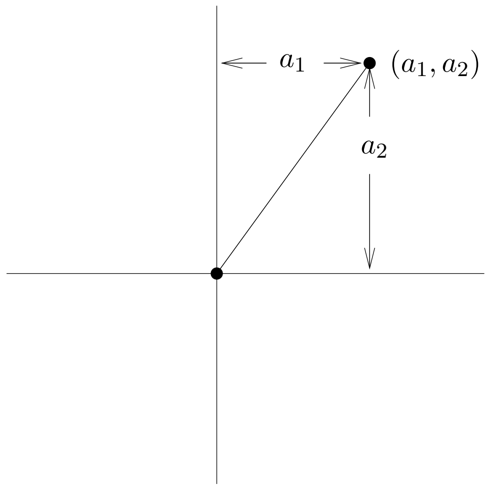


FIGURE 1. The segment from the origin is the hypotenuse of a right triangle with legs of size a_1 and a_2 and therefore the length of the hypotenuse is $\sqrt{a_1^2 + a_2^2}$.

Problem 11. Draw some pictures of when $\vec{a}$ is in the second and third quadrant (so that one the other or both of a_1 or a_2 is negative) and explain why the formula is still correct even when some of the components of $\vec{a}$ are negative. $\square$

If $\vec{a}$ and $\vec{b}$ are vectors the ***distance*** is

$$\operatorname{dist}(\vec{a}, \vec{b}) = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}.$$

This can again be motivated by the Pythagorean Theorem.

Problem 12. Draw a couple of picture showing why this is the correct definition of the distance between points based on the Pythagorean Theorem. $\square$

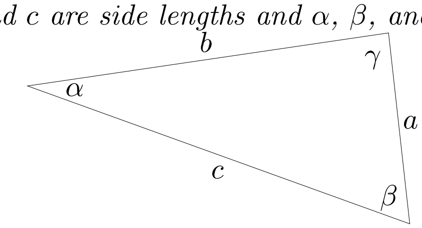
Proposition 14. *The distance between points is given by*

$$\operatorname{dist}(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\|.$$

Problem 13. Prove this. $\square$

Here is one way to motivate the definition of the inner product based on:

Theorem 15 (The Law of Cosines.)**.** *Let a triangle be labeled as in the figure where a, b, and c are side lengths and α, β, and γ are angles.*



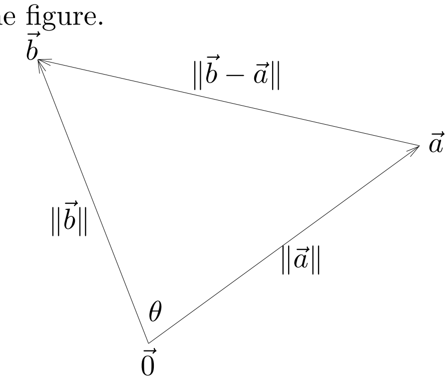
Then

$$\begin{aligned} a^2 &= b^2 + c^2 - 2bc\cos(\alpha) \\ b^2 &= a^2 + c^2 - 2ac\cos(\beta) \\ c^2 &= a^2 + b^2 - 2ab\cos(\gamma). \end{aligned}$$

□

Note in the case that $\gamma = 90°$, so that the triangle is a triangle, with legs a and b and hypotenuse c, then, as $\cos(\gamma) = 0$ this becomes $c^2 = a^2 + b^2$ which is just the Pythagorean theorem. Therefore the Law of Cosines can be thought of as a generalization of the Pythagorean to triangle that need not have a right angle.

Problem 14. This the correct version of what messed up in class. Given two vectors $\vec{a}$ and $\vec{b}$ consider the the following triangle which has vertices at $\vec{0}$, $\vec{a}$ and $\vec{b}$. Let θ be the angle between $\vec{a}$ and $\vec{b}$. Then side lengths of the triangle are as in the figure.



(a) Show $\|\vec{b} - \vec{a}\|^2 = \|\vec{a}\|^2 - 2\vec{a} \cdot \vec{b} + \|\vec{b}\|^2$.

(b) Explain why

$$\|\vec{b} - \vec{a}\|^2 = \|\vec{a}\|^2 + \|\vec{b}\|^2 - 2\|\vec{a}\|\|\vec{b}\|\cos\theta$$

(c) Use part (a) to show the equation of (b) simplifies to

$$\vec{a} \cdot \vec{b} = \|\vec{a}\|\|\vec{b}\|\cos\theta$$

(d) Use this to find the angle between the vector $\vec{a} = \langle 1, 2\rangle$ and $\vec{b} = \langle -2, 3\rangle$.

(e) If $\|\vec{a}\| = \|\vec{b}\|$ show that the two vector $\vec{v} = \vec{a} + \vec{b}$ and $\vec{w} = \vec{a} - \vec{b}$ are orthogonal. *Hint:* Two vectors are orthogonal if and only if their inner product is zero. □

Definition 16. A map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a ***rigid motion*** or ***isometry*** iff it preserves distances. That is for all $\vec{x}, \vec{y} \in \mathbb{R}^2$ we have

$$\|T(\vec{x}) - T(\vec{y})\| = \|\vec{x} - \vec{y}\|.$$

□

Let us give a few basic properties of rigid motions before we give examples.

Proposition 17. *Let S and T be rigid motions. Then the composition is a rigid motion. If T is a rigid motion and it has an inverse T^{-1}, then T^{-1} is also a rigid motion.*

Problem 15. Prove this. □

Definition 18. Let $\vec{a} \in \mathbb{R}^2$. Then the ***translation by*** $\vec{a}$ is the map

$$T_{\vec{a}}(\vec{x}) = \vec{x} + \vec{a}.$$

□

Proposition 19. *Each translation $T_{\vec{a}}$ is an rigid motion. In the case $\vec{a} = \vec{0}$ this translation is the identity map.*

Problem 16. Prove this. □

Definition 20. A 2×2 matrix P is ***orthogonal*** iff for all vectors $\vec{x}$ and $\vec{y}$ we have

$$(P\vec{x}) \cdot (P\vec{y}) = \vec{x} \cdot \vec{y}.$$

□

This terminology many seem a little strange at first, here part of an explanation. Write

$$P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$$

and let

$$P_1 = \begin{bmatrix} p_{11} \\ p_{21} \end{bmatrix}, \qquad P_2 = \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}$$

be the columns of P. Let

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

be the standard basis of $\mathbb{R}^2$.

Problem 17. We have done this before, but here it is again for review. Show

$$P\vec{e}_1 = P_1 \qquad P\vec{e}_2 = P_1.$$

That is $P\vec{e}_1$ is the first column of P and $P\vec{e}_2$ is the second column of P. □

Proposition 21. *If $P = [P_1, P_2]$ is a matrix with columns P_1 and P_2 and P is orthogonal, then P_1 and P_2 both have length 1 and P_1 and P_2 are orthogonal to each other. More explicitly*

$$\|P_1\| = \|P_2\| = 1, \qquad P_1 \cdot P_2 = 0.$$

□

Problem 18. Prove this. *Hint:* By the definition of P being orthogonal we have that for all vectors $\vec{x}$ and $\vec{y}$ that

$$(P\vec{x}) \cdot (P\vec{y}) = \vec{x} \cdot \vec{y}.$$

We now just need to make smart choices of $\vec{x}$ and $\vec{y}$. For example if $\vec{x} = \vec{e}_1$ and $\vec{y} = \vec{e}_2$, then we have

$$\|P_1\|^2 = P_1 \cdot P_1 = (P\vec{e}_1) \cdot (P\vec{e}_1) = \vec{e}_1 \cdot \vec{e}_1 = 1.$$

This $\|P_1\| = \sqrt{1} = 1$. The rest of the proof works along the same lines. □

The reason we care about orthogonal matrices is

Proposition 22. *Let P be an orthogonal matrix. Then for any vector $\vec{v}$ we have*

$$\|P\vec{v}\| = \|v\|$$

and therefore for all $\vec{x}$ and $\vec{y}$

$$\|P\vec{x} - P\vec{y}\| = \|\vec{x} - \vec{y}\|.$$

Therefore P is a rigid motion of $\mathbb{R}^2$. □

Problem 19. Prove this along the following lines. Let $P = [P_1, P_2]$ so that P_1 and P_2 are the columns. We have seen $\|P_1\| = \|P_2\| = 1$ and $P_1 \cdot P_2 = 0$. These are the only properties of P that we will use. Let

$$\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

(a) Show

$$Pv = v_1P_1 + v_2P_2.$$

Hint: One what to start is to use basic properties of matrix multiplication: $P\vec{v} = P(v_1\vec{e}_1 + v_2\vec{e}_2) = v_1P\vec{e}_1 + v_2P\vec{e}_2$ and use that $P\vec{e}_1 = P_1$ etc.

(b) Show $\|P\vec{v}\|^2 = v_1^2 + v_2^2 = \|\vec{v}\|^2$. *Hint:* One way to start is

$$\begin{aligned} \|P\vec{v}\|^2 &= (P\vec{v}) \cdot (P\vec{v}) \\ &= (v_1P_1 + v_2P_2) \cdot (v_1P_1 + v_2P_2) \\ &= v_1^2P_1 \cdot P_1 + 2v_1v_2P_1 \cdot P_2 + v_2P_2 \cdot P_2 \end{aligned}$$

and use what we know about $P_1 \cdot P_1$, $P_1 \cdot P_2$ etc.

(c) It should now be easy to see that $\|P\vec{v}\| = \|\vec{v}\|$.

(d) Finally given $\vec{x}$ and $\vec{y}$ we have by standard properties of matrices that

$$P\vec{x} - P\vec{y} = P(\vec{x} - \vec{y}).$$

Thus

$$\|P\vec{x} - P\vec{y}\| = \|P(\vec{x} - \vec{y})\|.$$

Let $\vec{v} = \vec{x} - \vec{y}$ and use parts (a), (b), and (c) to complete the proof. □

Proposition 23. *If P is a matrix such that for all vectors $\vec{v}$ we have*

$$\|P\vec{v}\| = \|\vec{v}\|$$

then P is orthogonal. That is for all $\vec{x}$ and $\vec{y}$ we have

$$(P\vec{x}) \cdot (P\vec{y}) = \vec{x} \cdot \vec{y}.$$

Problem 20. Prove this along the following lines.

(a) Show for any vectors $\vec{a}$ and $\vec{b}$ that

$$\vec{a} \cdot \vec{b} = \frac{1}{4}\left(\|\vec{a}+\vec{b}\|^2 - \|\vec{a}-\vec{b}\|^2\right).$$

(b) Use (a) to show and the basic property that $P(\vec{x}+\vec{y}) = P\vec{x} + P\vec{y}$ and $P(\vec{x}-\vec{y}) = P\vec{x} - P\vec{y}$ to show

$$(P\vec{x}) \cdot (P\vec{y}) = \frac{1}{4}\left(\|P(\vec{x}+\vec{y})\|^2 - \|P(\vec{x}-\vec{y})\|^2\right).$$

(c) Our hypothesis is that $\|P\vec{v}\| = \|\vec{v}\|$ for all $\vec{v}$. By first letting $\vec{v} = \vec{x}+\vec{y}$ and then $\vec{v} = \vec{x}-\vec{y}$ show

$$\|P(\vec{x}+\vec{y})\|^2 = \|\vec{x}+\vec{y}\|^2, \qquad \|P(\vec{x}-\vec{y})\|^2 = \|\vec{x}-\vec{y}\|^2$$

(d) Complete the proof. □

Theorem 24. *Let $P = [P_1, P_2]$ be a matrix with columns P_1 and P_2. Then the following are equivalent.*

(a) P is orthogonal. (That is $(P\vec{x}) \cdot (P\vec{y}) = \vec{x} \cdot \vec{y}$ for all $\vec{x}$ and $\vec{y}$.)
(b) The columns of P satisfy $\|P_1\| = \|P_2\| = 1$ and $P_1 \cdot P_2 = 0$.
(c) $\|P\vec{v}\| = \|\vec{v}\|$ for all $\vec{v}$.

Proof. Assume (a) holds. Then Proposition 21 implies (b) holds.

Assume (b) holds. Then we have seen in Problem 19 that (c) holds.

Assume (c) holds. Then Proposition 23 shows (a) holds.

We therefore have the implications

$$(a) \implies (b) \implies (c) \implies (a)$$

which shows the conditions are equivalent. □

Theorem 25. *Let P be a matrix. Then the map $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$T\vec{v} := P\vec{v}$$

is a rigid motion if and only if P is orthogonal.

Problem 21. Prove this. *Hint:* If P is orthogonal then it show not be hard to see that T is a rigid motion. Conversely if T is an isometry we have that $\|P\vec{x} - P\vec{y}\| = \|\vec{x} - \vec{y}\|$ for all $\vec{x}$ and $\vec{y}$. Let $\vec{y} = \vec{0}$ to get $\|P\vec{x}\| = \|\vec{x}\|$ and now use Theorem 24. □

We now give examples of some orthogonal maps.

Definition 26. For any number α let
$$R_\alpha = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$
This is the ***rotation by*** α about the origin. □

Proposition 27. *The rotation R_α is orthogonal.*

Problem 22. Prove this. □

To get a geometric picture of what R_α does take a point in the plane and write it in polar form:
$$\vec{v} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix}$$
Then
$$\begin{aligned} R_\alpha \vec{v} &= \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \\ &= \begin{bmatrix} r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) \\ r(-\sin\alpha\cos\theta + \cos\alpha\sin\theta) \end{bmatrix} \\ &= \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix} \end{aligned}$$
where we have used the addition formulas for sine and cosine. That is R_α maps the point $(r\cos\theta, r\sin\theta)$ to the point $(r\cos(\theta+\alpha), r\sin(\theta+\alpha))$. That is the argument of a point (that is the angle it makes with the positive x-axis) is increased by α.

Proposition 28. *If α and β are numbers*
$$R_\alpha \circ R_\beta = R_{\alpha+\beta}.$$
(Note for matrices composition and matrix multiplication are the same thing.)

Problem 23. Prove this. *Hint:* Use the formula for matrix multiplication and the addition formulas for the sine and cosine:
$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \\ \sin(\alpha+\beta) &= \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\alpha). \end{aligned}$$

Proposition 29. *The rotation R_0 is the identity map on $\mathbb{R}^2$. The inverse of R_α is $R_{-\alpha}$.*

Problem 24. Prove this. □

The matrix R_α gives the rotation about the point $\vec{0}$. But it is natural to do a rotation about any point of the plane. Here is how we can reduce the case of a rotation about a general point to a rotation about the origin by use of translations.

Problem 25. Let $\vec{a} \in \mathbb{R}^2$ and let α be a real number. Let
$$T = T_{\vec{a}} \circ R_\alpha \circ T_{-\vec{a}}.$$

(a) Show that T is a rigid motion. *Hint:* T is a composition of rigid motions so you can use Proposition 17.
(b) Show that $T(\vec{a}) = \vec{a}$. *Hint:* $T_{-\vec{a}}(\vec{a}) = \vec{0}$ and $R_\alpha(\vec{0}) = \vec{0}$.
(c) More generally show

$$T(\vec{a} + \vec{v}) = \vec{a} + R_\alpha \vec{v}.$$

(d) Using (c) draw some pictures and explain why T is a rotation of angle α about the point $\vec{a}$. $\square$