Modern Geometry Homework.

Our goal to to show that rigid motions are affine maps. That is if $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ satisfies

$$\|T\vec{a} - T\vec{b}\| = \|\vec{a} - \vec{b}\|$$

for all $\vec{a}, \vec{b} \in \mathbb{R}^2$, then

$$T((1-t)\vec{a} + t\vec{b}) = (1-t)T(\vec{a}) + tT(\vec{b}).$$

or all $\vec{a}, \vec{b} \in \mathbb{R}^2$ and all $t \in \mathbb{R}$.

We have shown

Proposition 1. *Let $\vec{a}$ and $\vec{b}$ be in $\mathbb{R}^2$ then a point $\vec{x}$ satisfies*

$$\vec{x} = (1-t)\vec{a} + t\vec{b}. \tag{1}$$

for some t with $0 < t < 1$ if and only if

$$|\|\vec{b} - \vec{x}\| + \|\vec{x} - \vec{a}\| = \|\vec{b} - \vec{a}\|. \tag{2}$$

If either of these two equivalent conditions hold then

$$t = \frac{\|\vec{x} - \vec{a}\|}{\|\vec{b} - \vec{a}\|}. \tag{3}$$

Proof. First assume that $\vec{x} = (1-t)\vec{a} + t\vec{b}$ with $0 < t < 1$. Then

$$\vec{b} - \vec{x} = \vec{b} - ((1-t)\vec{a} + t\vec{b}) = (1-t)(\vec{b} - \vec{a})$$

and

$$\vec{x} - \vec{a} = ((1-t)\vec{a} + t\vec{b}) - \vec{a} = t(\vec{b} - \vec{a}).$$

As $0 < t < 1$ both of t and $(1-t)$ are positive. Thus

$$\begin{aligned} \|\vec{b} - \vec{x}\| + \|\vec{x} - \vec{a}\| &= \|(1-t)(\vec{b} - \vec{a})\| + \|t(\vec{b} - \vec{a})\| \\ &= (1-t)\|\vec{b} - \vec{a}\| + t\|\vec{b} - \vec{a}\| \\ &= \|\vec{b} - \vec{a}\|. \end{aligned}$$

This shows that (2) holds.

Conversely if (2) holds, then

$$\begin{aligned} \|\vec{b} - \vec{a}\| &= \|(\vec{b} - \vec{+}(\vec{x} - \vec{a})\| \\ &\le \|(\vec{b} - \vec{x})\| + \|(\vec{x} - \vec{a})\| && \text{(By triangle inequality)} \\ &= \|\vec{b} - \vec{a}\| && \text{(As (2) holds).} \end{aligned}$$

This implies that equality holds in the triangle inequality. This means that $(\vec{b} - \vec{x})$ and $(\vec{x} - \vec{a})$ point in the same direction. Therefore there is a positive constant c such that

$$(\vec{b} - \vec{x}) = c(\vec{x} - \vec{a}).$$

We can solve this for $\vec{x}$ to get

$$\vec{x} = \frac{c}{1+c}\vec{a} + \frac{1}{1+c}\vec{b}.$$

Therefore if we set $t = \dfrac{1}{1+c}$ we have $0 < t < 1$ and $\vec{x} = (1-t)\vec{a} + t\vec{b}$ as required.

Finally if the conditions hold, then $x = (1-t)\vec{a} + t\vec{b}$. This can be rearranged as

$$t(\vec{b} - \vec{a}) = \vec{x} - \vec{a}.$$

Take the norms of both sides and divide by $\|1b - \vec{a}\|$ to get

$$t = \frac{\|\vec{x} - \vec{a}\|}{\|\vec{b} - \vec{a}\|}$$

which completes the proof. □

Proposition 2. *Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a rigid motion and $0 < t < 1$. Then for and $\vec{a}, \vec{b} \in \mathbb{R}^2$*

$$T((1-t)\vec{a} + t\vec{b}) = (1-t)T(\vec{a}) + tT(\vec{b}).$$

Proof. If $\vec{a} = \underline{1}$ this is clear. So assume $\vec{a} \neq \vec{b}$ and set

$$\vec{x} = (1-t)\vec{a} + t\vec{b}.$$

By the last Proposition we then have

$$t = \frac{\|\vec{x} - \vec{a}\|}{\|\vec{b} - \vec{a}\|}$$

and

$$\|\vec{b} - \vec{x}\| + \|\vec{x} - \vec{a}\| = \|\vec{b} - \vec{a}\|$$

Using this and that T is a rigid motion we have

$$\begin{aligned} \|T(\vec{b}) - T(\vec{x})\| + \|T(\vec{x}) - T(\vec{a})\| &= \|\vec{b} - \vec{x}\| + \|\vec{x} - \vec{a}\| \\ &= \|\vec{b} - \vec{a}\| \\ &= \|T(\vec{b}) - T(\vec{a})\| \end{aligned}$$

This by Proposition 1 (with $T(\vec{a})$ and $T(\vec{b})$ playing the part of $\vec{a}$ and $\vec{b}$ and s playing the part of t) there a $s \in \mathbb{R}$ with $0 < s < 1$ and

$$T(\vec{x}) = (1-s)T(\vec{a}) + sT(\vec{a})$$

where, again using that T is a rigid motion,

$$s = \frac{\|T(\vec{x}) - T(\vec{a}(\|}{\|T(\vec{b}) - T(\vec{a})\|} = \frac{\|\vec{x} - \vec{a}\|}{\|\vec{b} - \vec{a}\|} = t.$$

Putting this all together gives

$$T(1-t)\vec{a} + t\vec{b}) = T(\vec{x}) = (1-s)T(\vec{a}) + sT(\vec{a}) = (1-t)T(\vec{a}) + tT(\vec{b})$$

which is just what we wanted. □

Proposition 3. *Let* $\vec{a}$ *and* $\vec{b}$ *be in* $\mathbb{R}^2$ *then a point* $\vec{x}$ *satisfies*

$$\vec{x} = (1-t)\vec{a} + t\vec{b}. \tag{4}$$

for some t *with* $t > 1$ *if and only if*

$$|\|\vec{x} - \vec{b}\| + \|\vec{b} - \vec{a}\| = \|\vec{x} - \vec{a}\|. \tag{5}$$

If either of these two equivalent conditions hold then

$$t = \frac{\|\vec{x} - \vec{a}\|}{\|\vec{b} - \vec{a}\|}. \tag{6}$$

Problem 1. Prove this. *Hint:* Very much like the proof of Proposition 1. □

Proposition 4. *Let* $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a rigid motion and* $t > 1$*. Then for and* $\vec{a}, \vec{b} \in \mathbb{R}^2$

$$T((1-t)\vec{a} + t\vec{b}) = (1-t)T(\vec{a}) + tT(\vec{b}).$$

Problem 2. Prove this. *Hint:* Very much like the proof of Proposition 2. □