Review for test 3

Since the last test we have finished up the little we did on projective geometry. So I still expect you to still know the basics about projective geometry: The basic notations are ***points***, ***lines*** and ***incidence*** between points and lines.

Axiom 1 (First Axiom of Projective Geometry). *For any pair of distinct points there is unique line incident with these points.* □

Axiom 2 (Second Axiom of Projective Geometry). *For any pair of distinct lines there is unique point incident with these lines.* □

Axiom 3 (Thrid Axiom of Projective Geometry). *There are four points no three of which are incident with the same line.* □

The big result was that we used the affine version of Desargues' Theorem to prove the general (that is in projective planes) of the result. This is

Theorem 1 (Desargues' Theorem in Projective geometry). *Let $\triangle ABC$ and $\triangle A'B'C'$ be triangles in the projective plane. The the lines $\overleftrightarrow{AA'}$, $\overleftrightarrow{BB'}$ and $\overleftrightarrow{CC'}$ are concurrent if and only if the points $\overleftrightarrow{AB} \cap \overleftrightarrow{A'B'}$, $\overleftrightarrow{AC} \cap \overleftrightarrow{A'C'}$, and $\overleftrightarrow{BC} \cap \overleftrightarrow{B'C'}$ are colinear.* □

We then moved on to rigid motions of the plane, but did this by first looking at the rigid motions of the line. The ***distance*** between two points a and b on the line is $|a - b|$ and a function $T\colon \mathbb{R} \to \mathbb{R}$ is a rigid motions iff $|T(a) - T(b)| = |a - b|$ for all a and b.

Sample Problem. Find all the points x such that the x is twice as far from 1 as it is from 2. *Solution:* We want the points x such that $|x - 1| = 2|x - 2|$. To get rid of the absolute values square both sides to get

$$(x - 1)^2 = 4(x - 2)^2.$$

This is now a quadratic equation which we can solve by any number of methods. Here is one: rewrite as

$$4(x - 2)^2 - (x - 1)^2 = 0.$$

Use the identity $A^2 - B^2 = (A + B)(A - B)$ to get

$$(2(x - 2) + (x - 1))(2(x - 2) - (x - 1)) = 0,$$

that is

$$(3x - 5)(x - 3) = 0$$

so that $x = \frac{5}{3}, 3$. Since we squared an equation (and thus may have introduced extraneous roots) we should go back and check that both of these value of x work. They do. □

You should know Definitions 6 and 8 on Homework 5 and be able to do Problems 1–7 on Homework 5 related to these definitions.

You should also know the statement of Theorem 13 on Homework 5 and how to use to prove such things as showing that all rigid motions are affine maps.

Sample Problem. Find all rigid motions T of the line such that $T(1) = 2$. *Hint:* Split this into two cases. First find the translations that to this, and then fint the reflections that do it. $\square$

We then started working with rigid motions of the plane. You should understand the inner product $\vec{a} \cdot \vec{b}$ and its relation to the distance formula

$$\operatorname{dist}(\vec{a}, \vec{b}) = \|\vec{a} - \vec{b}\| = \sqrt{(\vec{a} - \vec{b}) \cdot (\vec{a} - \vec{b})}$$

In particular you should know that

$$\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$$

where θ is the angle between the vectors and how to prove this using the law of cosines (which means that you have to know the statement of the law of cosines).

Sample Problem. Use the identity $\vec{a} \cdot \vec{b} = \|\vec{a}\| \|\vec{b}\| \cos(\theta)$ to prove the law of cosines. $\square$

Sample Problem. Show that the set of points $\vec{x}$ that are an equal distance from the two points $\vec{a}$ and $\vec{b}$ is the line through $\frac{1}{2}(\vec{a} + \vec{b})$ and perpendicular to $\overleftrightarrow{\vec{a}\vec{b}}$. *Hint:* We did this in class, so look at your notes. $\square$

Sample Problem. Let $\vec{a}$, $\vec{b}$ and $\vec{c}$ be non collinear points. Show that there is exactly one point that is equal distance from all three of them. *Hint:* Let ℓ be the set of points that are an equal distance from $\vec{a}$ and $\vec{b}$. Let m be the set of points that are an equal distance from $\vec{b}$ and $\vec{b}$. Then by the last problem both ℓ and m are lines. Now explain that the set of points that are at an equal distance from all three points is $\ell \cap m$. $\square$

The then defined ***rigid motions*** of the plane. As to basic properties of these be able to prove Proposition 17 on Homework 5. We then defined translations, $T_{\vec{a}}$, and showed these are rigid motions.

We then defined ***rotations***, R_α, and their relationship to ***orthogonal matrices*** and you should know these definitions. Several of the properties of these were proven using the addition formulas for sin and cos. You should know these addition formulas and how to use them to prove things like Proposition 28 on Homework 5. Along these lines you should be able to do problems like Problems 15–25 on Homework 5.

Sample Problem. Let P and Q be orthogonal matrices. Show that the product PQ is also orthogonal.

Solution: The definition of a matrix, M, being orthogonal is that

$$M\vec{v} \cdot M\vec{w} = \vec{v} \cdot \vec{w}$$

for all $\vec{v}$ and $\vec{w}$. Thus we have

$$P\vec{v} \cdot P\vec{w} = \vec{v} \cdot \vec{w} \tag{1}$$

$$Q\vec{x} \cdot Q\vec{y} = \vec{x} \cdot \vec{y}. \tag{2}$$

Let $M = PQ$, then

$$\begin{aligned} M\vec{x} \cdot M\vec{y} &= P(Q\vec{x}) \cdot P(Q\vec{y}) \\ &= Q\vec{x} \cdot Q\vec{y} && \text{(By eqn (1) with } \vec{v} = Q\vec{x} \text{ and } \vec{w} = P\vec{y}) \\ &= \vec{x} \cdot \vec{y} && \text{(By eqn (2))} \end{aligned}$$

which is what we wanted to show. □

Sample Problem. Let M any invertible matrix and $T_{\vec{a}}$ the translation by $\vec{a}$. Show that $MT_{\vec{a}}M^{-1}$ is a translation.

Solution: This is just a definition chase.

$$\begin{aligned} MT_{\vec{a}}M^{-1}(x) &= M(T_{\vec{a}}(M^{-1}(x))) \\ &= M(M^{-1}(x) + \vec{a}) && \text{(Definition of } T_{\vec{a}}) \\ &= MM^{-1}(x) + M\vec{a} && \text{(Distributive prop. of matrix multiplication.)} \\ &= x + M\vec{a} && (MM^{-1} = I \text{ and } I\vec{x} = \vec{x}.) \end{aligned}$$

Thus $MT_{\vec{a}}M^{-1} = T_{M\vec{a}}$ which shows $MT_{\vec{a}}M^{-1}$ is the translation by $M\vec{a}$. □