Mathematics 172 Homework

We are about to start working with equations involving derivatives. So now is a good time to review some basic about them.

First recall that if

$$y = e^{rx}$$

where r is a constant, then the derivative is

$$y' = re^{rx}.$$

Thus if $f(t) = e^{4t}$ we have $f'(t) = 4e^{4t}$. If $P(t) = 100e^{.1t}$, then $P'(t) = 10e^{.1t}$. Here are some practice problems for you.

1. Find the following derivatives:

(a) $f(s) = 14e^{3s}$ **Answer:** $f'(s) = 42e^{3s}$

(b) $P(t) = 500e^{-.15s}$ **Answer:** $P'(t) = -75e^{-.15t}$

(c) $A(t) = 900e^{.05t}$ **Answer:** $A'(t) = 45e^{.05t}$.

Note if $P(t) = e^t$ that $P'(t) = e^t = P(t)$. That is in this case $P(t)$ is its own derivative. We also have that $P(0) = e^0 = 1$. Let us look at the converse of this. Assume that $P'(t) = P(t)$ and that $P(0) = 1$. Then we would like to say that $P(t) = e^t$. If this is true, then $e^{-t}P(t) = 1$ is constant. Recall that a function is constant exactly when its derivative is constant.

So set

$$f(t) = e^{-t}P(t).$$

We use the product rule to compute the derivative

$$f'(t) = (e^{-t})'P(t) + e^{-t}P'(t) = -e^{-t}P(t) + e^{-t}P(t) = 0$$

where we have used that $P'(t) = P(t)$. But $f'(t) = 0$ implies that $f(t)$ is a constant, say $f(t) = c$. We can find c by letting $t = 0$ and using that $P(0) = 1$.

$$c = f(0) = e^{-0}P(0) = (1)(1) = 1.$$

Therefore we have

$$f(t) = e^{-t}P(t) = c = 1.$$

Multiply both sides of this by e^t to get

$$P(t) = e^t.$$

Therefore we have shown:

Theorem 1. *If*

$$P'(t) = P(t) \qquad \textit{and} \qquad P(0) = 1$$

then

$$P(t) = e^t.$$

We can generalize this.

2. As a first try set

$$P(t) = 42e^{.5t}$$

and show

$$P'(t) = 21P(t) \qquad \text{and} \qquad P(0) = 42.$$

More generally:

3. Let r and P_0 be constants and set

$$P(t) = P_0 r^{rt}.$$

Show

$$P'(t) = rP(t) \qquad \text{and} \qquad P(0) = P_0.$$

Here is the result we will be using:

Theorem 2. *Let r be a constant and let $P(t)$ be a function that satisfies*

$$P'(t) = rP(t).$$

Then

$$P(t) = P(0)e^{rt}.$$

4. Verify this result. **Hint:** This is a variant on what we did above. Let

$$f(t) = e^{-rt}P(t).$$

Use the product rule to see that

$$f'(t) = -re^{-rt}P(t) + e^{-rt}P'(t).$$

Now use $P'(t) = P(t)$ so see that $f'(t) = 0$. Therefore $f(t)$ is constant. If $f(t)$ has the constant value c, then we have

$$e{-}rtP(t) = c$$

Solve this for $P(0)$ to get

$$P(t) - ce^{rt}.$$

Finally let $t = 0$ to see that $c = P(0)$.

5. Here are some problems on the use of the last result.

(a) Solve $P'(t) = .1P(t)$ and $P(0) = 400$. **Answer:** $P(t) - 400e^{.1t}$

(b) Solve $A'(s) = -.15A(s)$ and $A(0) = .82$ **Answer:** $A(s) = .82e^{-.15s}$

(c) Solve $N'(t) = 1.2N(t)$ and $N(0) = 1{,}200$.

Answer: $N(t) = 1{,}200e^{1.2t}$