Mathematics 172 Homework

We have started to look at ***differential equations***, also called ***rate equations***, which are equations that involve a function and its derivative. Most of the ones we will look at are of the form

$$\frac{dN}{dt} = f(N).$$

Here $f(N)$ is a function of N and it tells us how to compute the value of the derivative $N' = dN/dt$ given the value of N. So for example if the rate equation is

$$N' = 5N - 7$$

Then if $N(0) = 2$ we have

$$N'(0) = 5N(0) - 7 = 5(2) - 7 = 3$$

If $N(9) = 13$, then we can compute $N'(9)$ be

$$N'(9) = 5N(9) - 7 = 5(13) - 7 = 58.$$

1. Here is some practice in this circle of ideas. Let $P'(t) = .1P(t)(20-P(t))$.
(a) If $P(0) = 3$ find $P'(0)$. *Solution:* $P'(0) = .1(3) * (20 - 3) = 5.1$
(b) If $P(5) = 22$, find $P'(5)$. *Solution:* $P'(5) = .1(22)(20 - 22) = -4.4$.

2. Consider the the rate equation

$$\frac{dN}{dt} = .13N(n - 15)(N - 35)$$

3. For the rate equation

$$\frac{dP}{dt} = .1P(10 - P)$$

(a) Show that the constant function $P = 10$ is a solution. *Solution:* As $P = 10$ is a constant, its derivative is zero. That is $\frac{dP}{dt} = 0$ But $.1P(10-P) = .1(10)(10 - 10) = 0$. Therefore the constant $P = 10$ makes both sides of $\frac{dP}{dt} = .1P(10 - P)$ have the same value and so it is a solution.
(b) Show that the constant $P = 5$ is not a solution. *Solution:* As $P = 5$ is constant, it has $dP/dt = 0$. But when we plug $P = 5$ into $.1P(10 - P)$ we get $.1(5)(10 - 5) = 2.5 \neq 0$. So it is not a solution.

For a rate equation

$$\frac{dN}{dt} = f(N)$$

a constant N_* is an ***equilibrium solution*** if it is a solution to

$$f(N) = 0.$$

A calculation like the one to the last problem shows that if N_* is an equilibrium solution, then the constant function $N = N_*$ is a solution to the rate equation.

4. Find the equilibrium solutions to the following rate equations.
(a) $P' = -.3P + 30$. *Solution:* $P_* = 100$.

(b) $\frac{dN}{dt} = .7N(N-5)(N-20)$. *Solution:* There are three: $N_* = 0$, $N_* = 5$ and $N_* = 20$.

5. For the rate equation

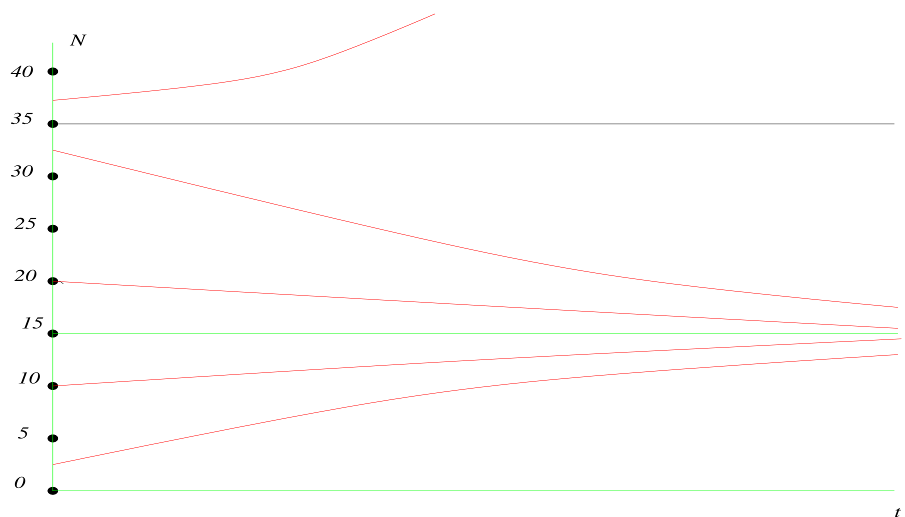
$$\frac{dN}{dt} = .7N(N-5)(N-20)$$

(a) What are the equilibrium solutions.

Answer: Set $\frac{dN}{dt} = .13N(N-15)(N-35) = 0$ and get $N = 0$, $N = 15$ and $N = 35$ as the equilibrium solution.

(b) Sketch the graphs of the solutions with the following initial values $N(0) = 2.5$, $N(0) = 10$, $N(0) = 20$, $N(0) = 32.5$, $N(0) = 37.5$

Answer: The equilibrium solutions are in green and the other required solutions are in red.

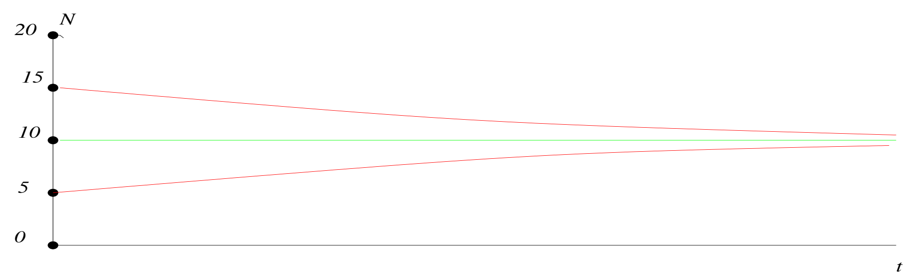


6. This time consider the rate equation

$$\frac{dP}{dt} = -2(P-10).$$

(a) Find the equilibrium solution. *Answer:* $P = 10$.

(b) Graph the equilibrium solution along with the solutions with initial values $P(0) = 5$ and $P(0) = 15$



(c) The figure makes it look like the solution could be exponential decay towards $P = 10$. So see if this is the case do a change of variable,

$$y = P - 10.$$

Show that y satisfies the rate equation

$$\frac{dy}{dt} = -2y.$$

Hint: $\dfrac{dy}{dt} = \dfrac{dP}{dt}$ as the derivative of 10 is zero.

(d) Show that y is given by

$$y(0) = y_0 e^{-2t}.$$

(e) So if we have the solution $P(t)$ of the original equation with $P(0) = 5$, then $y(0) = P(0) - 10 = 5 - 10 = -5$. Therefore

$$y(0) = y_0 e^{-2t} = -5e^{-2t}.$$

But $y = P - 10$ is implies $P = y + 10$. Therefore

$$P(t) = 10 - 5e^{-2t}.$$

Use the same change of variable $y = P - 10$ to show that the solution to $\frac{dP}{dt} = -2(P - 10)$ and $P(0) = 10$ is

$$P(t) = 5 + 5e^{-2t}.$$

7. Use the ideas of the last problem to find the solutions of

$$\frac{dN}{dt} = .5(N - 50)$$

with initial conditions $P(0) = 40$ and $P(0) = 65$. *Answer:* This time the change of variable is $y = N - 50$ and the equation for y is $y' = .5y$.

Answer: The solution for $N(0) = 40$ is $N(t) = 50 - 10e^{.5t}$ and the solution for $N(0) = 65$ is $N(t) = 50 + 15e^{.5t}$.