Mathematics 172 Homework

We look at finding the stable age distribution. Assume that we have a Leslie matrix

$$L = \begin{bmatrix} f_1 & f_2 & f_3 \\ p_1 & 0 & 0 \\ 0 & p_2 & 0 \end{bmatrix}$$

and that

$$\vec{n}(t) = \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}$$

gives the numbers in each of the three stages. What the Leslie matrix tells us is that the growth from one year to the next is given by

$$n_1(t+1) = f_1 n_1(t) + f_2 n_2(t) + f_3 n_3(t) \tag{1}$$

$$n_2(t+1) = p_1 n_1(t) \tag{2}$$

$$n_3(t+1) = p_2 n_2(t). \tag{3}$$

Now let

$$\begin{aligned} N(t) &= n_1(t) +_2 (t) + n_3(t) &&= \text{Total number of in year } t \\ N(t+1) &= n_1(t+1) +_2 (t+1) + n_3(t+1) &&= \text{Total number of in year } t+1 \end{aligned}$$

Then we have

$$\begin{aligned} \text{Proportion in sage 1 in year } t &= \frac{n_1(t)}{N(t)} \\ \text{Proportion in sage 2 in year } t &= \frac{n_2(t)}{N(t)} \\ \text{Proportion in sage 3 in year } t &= \frac{n_3(t)}{N(t)} \end{aligned}$$

$$\begin{aligned} \text{Proportion in sage 1 in year } t+1 &= \frac{n_1(t+1)}{N(t+1)} \\ \text{Proportion in sage 2 in year } t+1 &= \frac{n_2(t+1)}{N(t+1)} \\ \text{Proportion in sage 3 in year } t+1 &= \frac{n_3(t+1)}{N(t+1)} \end{aligned}$$

We now assume that we are at the stable age distribution. Then these proportions stay the same from year to year and thus

$$\frac{n_1(t+1)}{N(t+1)} = \frac{n_1(t)}{N(t)}$$
$$\frac{n_2(t+1)}{N(t+1)} = \frac{n_2(t)}{N(t)}$$
$$\frac{n_3(t+1)}{N(t+1)} = \frac{n_3(t)}{N(t)}$$

Multiple both sides of these equations by $N(t+1)$ to get

$$n_1(t+1) = \frac{N(t+1)}{N(t)} n_1(t)$$
$$n_2(t+1) = \frac{N(t+1)}{N(t)} n_2(t)$$
$$n_3(t+1) = \frac{N(t+1)}{N(t)} n_3(t).$$

If we simplify notation by letting

$$\lambda = \frac{N(t+1)}{N(t)}$$

these equations become

$$n_1(t+1) = \lambda n_1(t)$$
$$n_2(t+1) = \lambda n_2(t)$$
$$n_3(t+1) = \lambda n_3(t).$$

But we also have the formulas (1), (2), and (3) using these we find

(4) $$\lambda n_1(t) = f_1 n_1(t) + f_2 n_2(t) + f_3 n_3(t)$$
(5) $$n\lambda n_2(t) = p_1 n_1(t)$$
(6) $$n\lambda n_3(t) = p_2 n_2(t).$$

Using equation (5) we find

$$n_2(t) = \frac{p_1 n_1(t)}{\lambda}.$$

Now use this in equation (6) to get

$$n_3(t) = \frac{p_2 n_2(t)}{\lambda} = \frac{p_1 p_2 n_1(t)}{\lambda^2}$$

Use these formulas for $n_1(t)$ and $n_2(t)$ in equation (4) to get

$$\lambda n_1(t) = f_1 n_1(t) + \frac{p_1 f_2 n_1(t)}{\lambda} + \frac{p_1 p_2 f_3 n_1(t)}{\lambda^2}$$

Note that every term in this equation contains a factor of $n_1(t)$ and therefore these can be cancelled out of the equation. Then dividing the result by λ gives

$$1 = \frac{f_1}{\lambda} + \frac{p_1 f_2}{\lambda^2} + \frac{p_1 p_2 f_3}{\lambda^3}$$

This is the ***Euler-Lotka*** equation and it gives an equation for λ, which is not hard to solve using a calculator. Once we have found λ, we have

$$\begin{aligned} n_1(t) &= n_1(t) \\ n_2(t) &= \frac{p_1 n_1(t)}{\lambda} \\ n_3(t) &= \frac{p_1 p_2 n_1(t)}{\lambda^2} \end{aligned}$$

Then the total number is

$$N(t) = n_1(t) + n_2(t) + n_3(t) = n_1(t)\left(1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}\right).$$

It turns out that $n_1(t)$ will cancel out of our final equations, so we can let it be what ever makes the calculations easiest. The usual choice is $n_1(t) = 1$. Then we have

$$\begin{aligned} &n_1(t) 1 \\ &n_2(t) = \frac{p_1}{\lambda} \\ &n_3(t) = \frac{p_1 p_2}{\lambda^2} \\ &N(t) = n_1(t) + n_2(t) + n_3(t). \end{aligned}$$

and the stable age distribution is

$$\begin{aligned} \text{Proportion in Stage 1} &= \frac{n_1(t)}{N(t)} \\ \text{Proportion in Stage 2} &= \frac{n_2(t)}{N(t)} \\ \text{Proportion in Stage 3} &= \frac{n_3(t)}{N(t)} \end{aligned}$$

Here are some problems practice these ideas.

1. For the Leslie matrix

$$L = \begin{bmatrix} 0.0 & 2.4 & 16.0 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \end{bmatrix}$$

Find the Euler-Lotka equation, the value of λ and the stable age distribution.

Solution: The Euler-Lotka equation is

$$1 = \frac{.24}{\lambda^2} + \frac{.8}{\lambda^3}.$$

The value of λ is $\lambda = 1.01427034974610$ is

$$\text{The proportion in Sage } 1 = 0.8717$$
$$\text{The proportion in Sage } 2 = 0.0859$$
$$\text{The proportion in Sage } 3 = 0.0424$$

2. For the Leslie matrix

$$L = \begin{bmatrix} 0.0 & 1.35 & 18.0 \\ 0.2 & 0.0 & 0.0 \\ 0.0 & 0.25 & 0.0 \end{bmatrix}$$

Find the Euler-Lotka equation, the value of λ and the stable age distribution.

Solution: The Euler-Lotka equation is

$$1 = \frac{.27}{\lambda^2} + \frac{.9}{\lambda^3}$$

The value of λ is $\lambda = 1.05844437184970$ and the stable age distribution is

$$\text{The proportion in Sage } 1 = 0.8106$$
$$\text{The proportion in Sage } 2 = 0.1532$$
$$\text{The proportion in Sage } 3 = 0.0362$$

3. As a check for the Leslie matrix of the last problem let

$$\vec{n}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

and find $\vec{n}(50)$ and the age distribution for $t = 50$. Is it close to what the last problem predicts?

Solution:

$$\vec{n}(50) = \begin{bmatrix} 619.5618 \\ 117.4591 \\ 27.6617 \end{bmatrix}.$$

and the age distribution for $\vec{n}(50)$ is

$$\text{The proportion in Sage } 1 = 0.8102$$
$$\text{The proportion in Sage } 2 = 0.1536$$
$$\text{The proportion in Sage } 3 = 0.0362$$