Mathematics 172 Homework

We have seen that for an organism that reproduces once a year and has a per capita growth rate of r that

$$\Delta P = rP$$

where

$$P = P_t = \text{Number of individuals alive in year } t.$$

and

$$\Delta P = P_{t+1} - P_t$$

is the change in the population size from year t to year $t+1$. This can be rewritten as

$$P_{t+1} = (1+r)P_t = \lambda P_t$$

where $\lambda = 1 + r$ is the ***finite growth rate***. This says that to get the size of the population in year $t+1$ we just multiply the population size in year t by $\lambda = 1 + r$.

Let

$$P_0 = \text{Size of initial population.}$$

Then we can get the size of the populations for the first several years:

$$\begin{aligned}
P_1 &= \lambda P_0 = P_0\lambda \\
P_2 &= \lambda P_1 = \lambda P_0\lambda = P_0\lambda^2 \\
P_3 &= \lambda P_2 = \lambda P_0\lambda^2 = P_0\lambda^3 \\
P_4 &= \lambda P_3 = \lambda P_0\lambda^3 = P_0\lambda^4 \\
P_5 &= \lambda P_4 = \lambda P_0\lambda^4 = P_0\lambda^5 \\
P_6 &= \lambda P_5 = \lambda P_0\lambda^5 = P_0\lambda^6 \\
P_7 &= \lambda P_6 = \lambda P_0\lambda^6 = P_0\lambda^7
\end{aligned}$$

At this point you see the pattern:

$$P_t = P_0\lambda^t.$$

Example 1. Assume that 20 rats are introduced on an island and that the per capita growth rate of the rats is $r = 4.5$ rats/rat. Then what is a formula for the number of rats after t years? How many rats are there in five years? How many long until there are a million rats?

Solution: In this case we have $P_0 = 20$ and $\lambda = 1 + r = 5.5$ Therefore the formula for the number in t years is

$$P_t = P_0\lambda^t = 20(5.5)^t.$$

Therefore after ten years, that is $t = 5$, the number of rats is

$$P_{10} = 10(5.5)^5 = 100{,}656.875 \text{ rats.}$$

(This can be rounded to the nearest rat to get $P_5 \approx 100{,}657$ rats. Finally to see how long until a million rats we want to solve:

$$20(5.5)^t = 1{,}000{,}000 = 10^6,$$

This gives

$$(5.5)^t = (10^6)/20$$

and therefore

$$t \ln(5.5) = \ln\left(10^6/20\right)$$

and thus

$$t = \ln\left(10^6/20\right) / \ln(5.5) = 6.3468487 \text{ years.}$$

1. In a new dorm someone introduces 13 roaches. Let P_t be the number of roaches t weeks later. Assume that the per capita growth rate of the roaches is 2.3 roaches/roach each week. (a) Give a formula for the number of roaches after t weeks. (b) How long until there are a million roaches in the dorm? (c) How many are there after a year (which to make things simple we take to be 52 weeks.)

Solution: (a) $P_t = 13(3.3)^t$. (b) Solve $13(3.3)^t = 10^6$ to get $t = 9.423$ weeks, (c) $P_{52} = 13(3.3)^{52} = 1.1931 \times 10^{28}$ roaches.

Example 2. Assume that 25 sunflowers are introduced into a large field. Sunflowers are annuals. Assume that after three years there are 40 sunflowers in the field. Use this information to find a formula for the number of sunflowers after t years and use this to predict the number that will be in the field after 10 years.

Solution: The number is $P_t = P_0\lambda^t$. We know that $P_0 = 25$, but we still have to find λ. We have

$$P_3 = 25\lambda^3 = 40.$$

This leads to

$$\lambda^3 = (40/25)$$

and thus

$$\lambda = (40/25)^{1/3} = 1.1696\ .$$

Therefore

$$P_t = 25(1.1696)^t$$

Therefore the number after 10 years is

$$P_{10} = 25 * (1.1696)^{10} = 119.77 \text{ sunflowers.}$$

2. Chickweed (Stellaria media) is an annual plant that is considered a weed. Assume that 9 chickweeds are introduced into a large park and that 5 years later there are 100 chickweeds in the park. (a) Find a formula for the both the number of chickweeds in the park after t years and for r the per capita growth rate. (b) How many chick weeks are there in the park after 10 years? (b) How long until there are 10,000 chickweeds in the park?

Solution: (a) First show that $\lambda = 1.61864$ and therefore $P_t = 9(1.61864)^t$. Also $r = \lambda - 1 = .61864$. (b) $P_{10} = 9(1.61864)^{10} = 1{,}111.1$ chickweeds. (c) $t = 9.7812$ years.

Photo of Chickweed.