Mathematics 172 Homework

We saw in class today that if we assume that a population grows with an intrinsic growth rate of r (when the population size is small) but there is a ***carrying capacity***, K, where for populations of size larger than K the per capita growth rate becomes negative, that a reasonable model for the growth of the population size is

$$P_{t+1} = P_t + rP_t\left(1 - \frac{P_t}{K}\right).$$

More generally we will see models where the growth is determined by

$$P_{t+1} = f(P_t)$$

where $f(P)$ is some function of the population. Such an equation is called a ***difference equation*** and what tells us is that if we know the size of the population, P_t, is the year, t, then the size of the population the next year is $P_{t+1} = f(P_t)$. Here is an example. Assume the difference equation is

$$P_{t+1} = P_t + .3P_t\left(1 - \frac{P_t}{100}\right).$$

Assume that we know that in some year the population is $P_0 = 80$. Then the population the next year is

$$P_1 = P_0 + .3P_0\left(1 - \frac{P_0}{100}\right) = 80 + .3P_t\left(1 - \frac{80}{100}\right) = 84.8.$$

And the population the year after that is

$$P_1 = P_0 + .3P_1\left(1 - \frac{P_1}{100}\right) = 84.8 + .3P_t\left(1 - \frac{84.8}{100}\right) = 88.666880.$$

1. Show that for this difference equation we have

$$\begin{aligned}
P_0 &= 80.000\\
P_1 &= 84.800\\
P_2 &= 88.667\\
P_3 &= 91.681\\
P_4 &= 93.969\\
P_5 &= 95.670
\end{aligned}$$

2. Show that for the diffence equation

$$N_{t+1} = \frac{20N_t}{1 + .2N_t^2}$$

and $N_0 = 5$ that

$$P_0 = 5.000$$
$$P_1 = 16.667$$
$$P_2 = 5.894$$
$$P_3 = 14.832$$
$$P_4 = 6.592$$

3. Let $P_{t+1} = f(P_t)$ be a difference equation and let P_* be a number such that $f(P_*) = P_*$. Such points are called ***equilibrium*** points of the equation. Then show that if $P_0 = P_*$ that $P_t = P_*$ for all t. *Solution:* Here is the idea. $P_1 = f(P_0) = f(P_*) = P_*$. Therefore $P_1 = P_*$. Now $P_2 = f(P_1) = f(P_*) = P_*$. Now you use similar calculations to show that $P_3 = P_4 = P_5 = P_*$. This pattern continues for all t.

4. Find all the equilibrium points of

$$P_{t+1} = P_t + .3P_t\left(1 - \frac{P_t}{200}\right).$$

Solution: This is no more that a complicated way of asking us to find the solutions to the equation

$$P = P + .3P\left(1 - \frac{P}{200}\right).$$

Subtracting P from both sides gives

$$0 = .3P\left(1 - \frac{P}{200}\right).$$

and now a bit of algebra shows that the only equilibrium points are

$$P_* = 0 \qquad \text{and} \qquad P_* = 200$$