Mathematics 172 Homework

Read over the last homework and make sure you understand about equilibrium points. One of the things we mentioned in class is that if N_* is an equilibrium point of

$$N_{t+1} = f(N_t)$$

then it is stable if $|f'(N_*)| < 1$ and unstable if $|f'(N_*)| > 1$. We will explain why this holds in class.

1. Let $r, K > 0$. Then the discrete logistic with pre capita growth rate of r and carrying capacity K is

$$N_{t+1} = N_t + rN_t\left(1 - \frac{N_t}{K}\right) = f(N)$$

where

$$f(N) = N + rN\left(1 - \frac{N}{K}\right) f(N)$$

We wish to find the equilibrium points and see if they are stable.

(a) We first look at the special case where $r = .2$ and $K = 100$. Find the equilibrium points and determine if they are stable. *Solution:* In this case we wish to solve

$$f(N) = N + .2N\left(1 - \frac{N}{100}\right) = N.$$

This reduces to

$$.2N\left(1 - \frac{N}{100}\right) = 0$$

and we see the equilibrium are

$$N_* = 0, 100.$$

Now compute the derivative of f. To start it is a bit easier if we first rewrite f a bit.

$$f(N) = N + .2N - \frac{.2N^2}{100}.$$

This

$$f'(N) = 1 + .2 - \frac{.4N}{100}.$$

At $N_* = 0$ we have

$$f'(0) = 1 + .2 - \frac{.4(0)}{100} = 1.2 > 1$$

and therefore $N_* = 0$ is unstable. At $N_* = 100$ we have

$$f'(100) = 100 + .2 - \frac{.4(100)}{100} = .8$$

which shows that this point is also stable.

(b) Now do the general case where

$$f(N) = N + rN\left(1 - \frac{N}{K}\right) f(N)$$

Solution: The equilibrium points are $N_* = 0$ and $N_* = K$ A calculation like the ones done above yield that

$$f'(0) = 1 + r > 1$$

and so for the logistic equation $N_* = 0$ is always unstable. We also have

$$f'(K) = 1 - 2r$$

This in this case $N_* = K$ is stable when $0 < r < 2$ (which implies $|1-2r| < 1$ and it is unstable when $2 < r$ (which implies $|1 - 2r| > 1$).

2. (This is a bit of a challenging problem if you are not up in use some of the more advanced features on your calculator.) Let

$$f(P) = \frac{5 + 20P}{1 + P^2}$$

and consider the discrete dynamical

$$P_{t+1} = f(P_t).$$

(a) Graph $y = f(x)$ and $y = x$ for with $0 \leq x \leq 10$ and use the calculator to find the where these graphs intersect. *Solution:* There is only one point of intersection and it is $P_* = 4.48495684796404$.

(b) Use the calculator to find $f'(P_*)$. *Solution:* $f'(P_*) = -0.958078670794696$.

(c) Is P_* stable or unstable? *Solution:* since $|f'(P_*)| = 0.958078670794696 < 1$ the point is stable.