

Mathematics 172 Homework, February 7, 2018.

Read over the last homework and make sure you understand about equilibrium points. One of the things we mentioned in class is that if N_* is an equilibrium point of

$$N_{t+1} = f(N_t)$$

then it is stable if $|f'(N_*)| < 1$ and unstable if $|f'(N_*)| > 1$. We will explain why this holds in class.

Problem 1. Let $r, K > 0$. Then the discrete logistic with pre capita growth rate of r and carrying capacity K is

$$N_{t+1} = N_t + rN_t \left(1 - \frac{N_t}{K}\right) = f(N)$$

where

$$f(N) = N + rN \left(1 - \frac{N}{K}\right)$$

We wish to find the equilibrium points and see if they are stable.

(a) We first look at the special case where $r = .2$ and $K = 100$. Find the equilibrium points and determine if they are stable. *Solution:* In this case we wish to solve

$$f(N) = N + .2N \left(1 - \frac{N}{100}\right) = N.$$

This reduces to

$$.2N \left(1 - \frac{N}{100}\right) = 0$$

and we see the equilibrium are

$$N_* = 0, 100.$$

Now compute the derivative of f . To start it is a bit easier if we first rewrite f a bit.

$$f(N) = N + .2N - \frac{.2N^2}{100}.$$

This

$$f'(N) = 1 + .2 - \frac{.4N}{100}.$$

At $N_* = 0$ we have

$$f'(0) = 1 + .2 - \frac{.4(0)}{100} = 1.2 > 1$$

and therefore $N_* = 0$ is unstable. At $N_* = 100$ we have

$$f'(100) = 100 + .2 - \frac{.4(100)}{100} = .8$$

which shows that this point is also stable.

(b) Now do the general case where

$$f(N) = N + rN \left(1 - \frac{N}{K}\right)$$

Solution: The equilibrium points are $N_* = 0$ and $N_* = K$. A calculation like the ones done above yield that

$$f'(0) = 1 + r > 1$$

and so for the logistic equation $N_* = 0$ is always unstable. We also have

$$f'(K) = 1 - 2r$$

This in this case $N_* = K$ is stable when $0 < r < 2$ (which implies $|1 - 2r| < 1$) and it is unstable when $2 < r$ (which implies $|1 - 2r| > 1$). \square

Putting this all together gives

Theorem 1. *For the discrete logistic equation with per capita growth rate r and carrying capacity K ,*

$$N_{t+1} = N_t + rN_t \left(1 - \frac{N_t}{K}\right)$$

there are two equilibrium points $N_ = 0$ and $N_* = K$.*

- $N_* = 0$ is always unstable for the discrete logistic.
- $N_* = K$ is stable for $0 < r < 2$ and unstable for $r > 2$.

Here are some problem for finding equilibrium points in discrete dynamical and determining if they are stable using your calculator.

Problem 2. For the dynamical system

$$P_{t+1} = P_t + .3P_t \left(1 - \frac{P_t^{1.5}}{100}\right)$$

Find the equilibrium points and determine if they are stable.

Solution: First enter

`\Y1=X+.3X(1-X^(1.5)/100)`

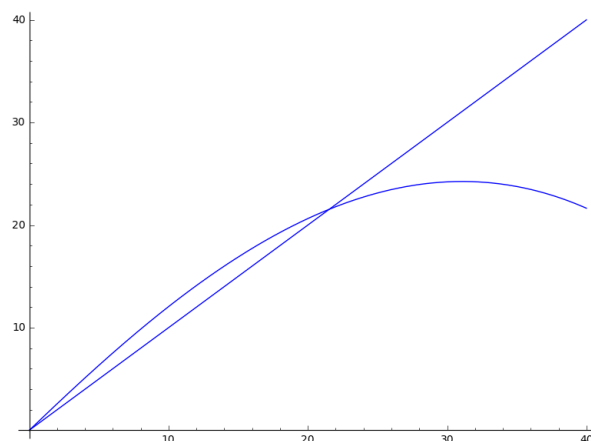
`\Y2=X`

And by some trial and error I found that good size for the window is

`Xmin=0`

`Xmax=40`

Now plot by using ZOOM and then 0:ZoomFit. The result should look something like:



From the picture we see that there are two equilibrium points.

To find them use **2nd CALC 5:intesect** It will now ask your **First curve**. Just hit **ENTER**. It will now ask **Second curve**. Again just hit **ENTER**. The next (and last) question is **Guess?**. This time move the cursor to be as close as possible to the intersection point you want to find and hit **ENTER**. If you moved the cursor to 0 (which is clearly an equilibrium point from the picture) we get that $x=0$ and $Y=0$ is an intersection point. To find if this is stable we hit **2nd CALC 6:dy/dx** which will give us the value of the derivative of the Y_1 curve at the current x value, which is $x = 0$. This will tell us that $f'(0) = 1.3$. As $|f'(0)| = 1.3 > 1$ the point $P_* = 0$ is unstable.

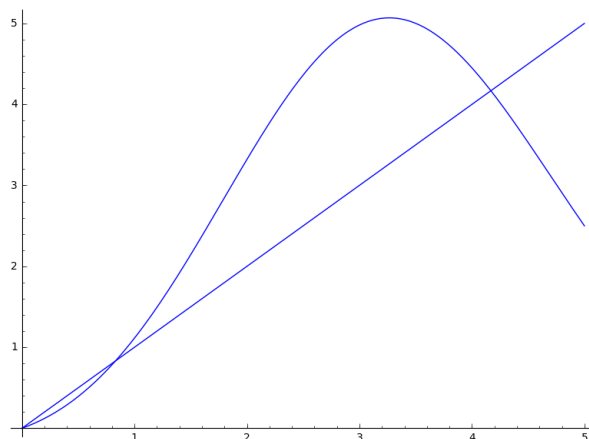
To find the second equilibrium point go through the steps above with the difference that when you are asked **Guess?**, you move the cursor to get as close as you can to the other point of intersection. This time you will get that $X=21.544347$ and that Y has the same value. Therefore $P_* = 21.544347$ is the second equilibrium point. Now use the calculator to find the value of $f'(21.544347) = .55$ so this point is stable.

Problem 3. For the dynamical system

$$N_{t+1} = .5N_t e^{N_t - .2N_t^2}$$

(a) Plot $y = f(x) = .5xe^{x-.2x^2}$ and $y = x$ on your calculator for $0 \leq x \leq 5$.

Solution: Your picture should look like:



(b) So we see there are three equilibrium points. Now find them and determine if they are stable or unstable.

Solution: The first is $P_* = 0$. As $f'(0) = .500$ this one is stable.

The second is $P_* = .83138857$ and at this point $f'(.83138857) = 1.5549057$ so this one is unstable.

The third is $P_* = 4.1686114$ and at this point $f'(4.1686114) = -1.759351$ so this one is unstable.

Problem 4. Let

$$f(P) = \frac{5 + 20P}{1 + P^2}$$

and consider the discrete dynamical

$$P_{t+1} = f(P_t).$$

(a) Graph $y = f(x)$ and $y = x$ for with $0 \leq x \leq 10$ and use the calculator to find the where these graphs intersect. *Solution:* There is only one point of intersection and it is $P_* = 4.48495684796404$.

(b) Use the calculator to find $f'(P_*)$. *Solution:* $f'(P_*) = -0.958078670794696$.

(c) Is P_* stable or unstable? *Solution:* since $|f'(P_*)| = 0.958078670794696 < 1$ the point is stable.