Mathematics 172 Homework, February 17, 2018.

Problem 1. Here are some practice problems on derivatives of the type that may come up in class. We will mostly be working with exponential functions, but polynomials and logarithms will also come up. It these exercises

$$x, y, z, t, P, N \quad \text{are variables,}$$

and

$$a, b, c, r, N_0, P_0 \quad \text{are constants.}$$

Verify the following

$$\begin{aligned}
y &= e^x & y' &= e^x \\
y &= 6e^{3x} & y' &= 18e^{3x} \\
P(t) &= 300e^{.12} & P'(t) &= 36e^{.12t} \\
N(t) &= 5{,}000e^{.2t} & \frac{dN}{dt} &= 1{,}000e^{.2t} \\
P(t) &= P_0e^{rt} & P'(t) &= rP_0e^{rt} \\
N(t) &= N_0e^{at} & \frac{dN}{dt} &= aN_0e^{at} \\
y &= xe^x & y' &= e^x + xe^x \\
f(t) &= 2t^4e^{5t} & f'(t) &= 10t^4e^{5t} + 8t^3e^{5t} \\
w &= z^2e^{z^3} & \frac{dw}{dz} &= 3z^3e^{z^3} + 2ze^{z^3}
\end{aligned}$$

The main result we saw in class on Friday was that if $P(t)$ is a function that satisfies

$$P'(t) = rP(t)$$

where r is a constant, then

$$P(t) = P(0)e^{rt}.$$

This can be written is slightly different notation and some of this notation is likely what you will see in other classes. Here are some restatements.

$$P' = rP \quad \text{implies} \quad P = P_0e^{rt}.$$

$$\frac{dP}{dt} = rP \quad \text{implies} \quad P(t) = P_0e^{rt}.$$

Of course there is nothing special about using P.

$$N' = rN \quad \text{implies} \quad N = N_0e^{rt}.$$

and

$$\frac{dA}{dt} = rA \quad \text{implies} \quad A(t) = A_0e^{rt}.$$

Problem 2. If $P'(t) = .3P(t)$ and $P(0) = 50$, (a) Give a formula for $P(t)$

(b) What is $P(5.3)$.

(c) When is $P(t) = 10{,}000$?

Solution: (a) $P(t) = 50e^{.3t}$. (b) $P(5.3) = 50e^{.3(5.3)} = 245.187$. (c) Solve $50e^{.3t} = 10,000$ to get $t = 17.661$. $\square$

Problem 3. If

$$\frac{dA}{dt} = -.05A \quad \text{and} \quad A(0) = 4.2$$

(a) Give a formula for $A(t)$.
(b) What is the half life.
(c) Where is there only 1% of the original amount left?

Solution: (a) $A(t) = 4.2e^{-.05t}$. (b) Solve $A(t) = (1/2)A(0)$, that is $4.2e^{-.05t} = (.5)4.2$, which is equivalent to $e^{-.05t} = .5$. This gives $t = 13.863$ (c) This time solve $A(t) = .01A(0)$, which is equivalent to $e^{-.05t} = .01$. The answer is $t = 92.103$. $\square$

We will start using differential equations in our models. These are equations that involve derivatives. Most of the ones we will see in the near future are of the form

$$N' = f(N)$$

or $P' = f(P)$ or $\dfrac{dP}{dt} = f(P)$ which tell us how to compute the derivative if we know the function value. For example if

$$P' = .5P(20 - P)$$

then if $P(0) = 5$ then

$$P'(0) = .5P(0)(20 - P(0)) = .5(5)(20 - 5) = 37.5$$

And for this same equation if $P(5.6) = 22.1$ then

$$P'(5.6) = .5P(5.6)(20 - P(5.6)) = .5(22.1)(20 - 22.1) = -23.205$$

Problem 4. Assume $N(t)$ satisfies

$$\frac{dN}{dt} = -.05N + 100$$

(a) If $N(0) = 90$ what is $N'(0)$. *Solution:* $N'(0) = -.05N(0) + 100 = -.05(90) + 100 = 95.5$.

(b) If $N(12) = 200$ what is $N'(12)$? *Solution:* $N'(12) = -.05N(12) + 100 = -.05(200) + 100 = 90$

(c) What value(s) of N make the derivative $= 0$? *Solution:* Solve the equation $N' = -.05N + 100 = 0$ to get

$$N = \frac{100}{.05} = 2{,}000.$$