Mathematics 172 Homework, February 24, 2018.

Let $P(t)$ be the size of a population at time t. Then

$$P' = \frac{dP}{dt}$$

is the rate of change of the population with respect to time. Then

$$\frac{1}{P}\frac{dP}{dt} = \frac{P'}{P}$$

is the rate of change per unit of P. For example if $P(t)$ is the number of grams of bacteria in a culture t hours after it is started, then

$$\frac{P'}{P} = \text{rate that each gram of bacteria is contributing to growth.}$$

If there is no constraints on the growth, then it is reasonable to expect this to be constant. Let this constant be r. Then we have

$$\frac{P'}{P} = r$$

which can be rewritten as

$$P' = rP$$

which is out basic model for exponential growth and r is the ***intrinsic growth rate***.

For small populations in a large environment and for short times this is a good model. But in the long term, that is when the time t is large, it is not realistic. If $r < 0$ then this model predicts that $P(t)$ will decrease to zero, that is the population will die off. If $r > 0$, then we have exponential growth and the population becomes infinite, which is impossible.

A reasonable way around this is to assume the intrinsic growth depends on the population size, and that the larger the population size smaller the growth rate becomes due to crowding. So we assume that

$$\frac{P'}{P} = f(P).$$

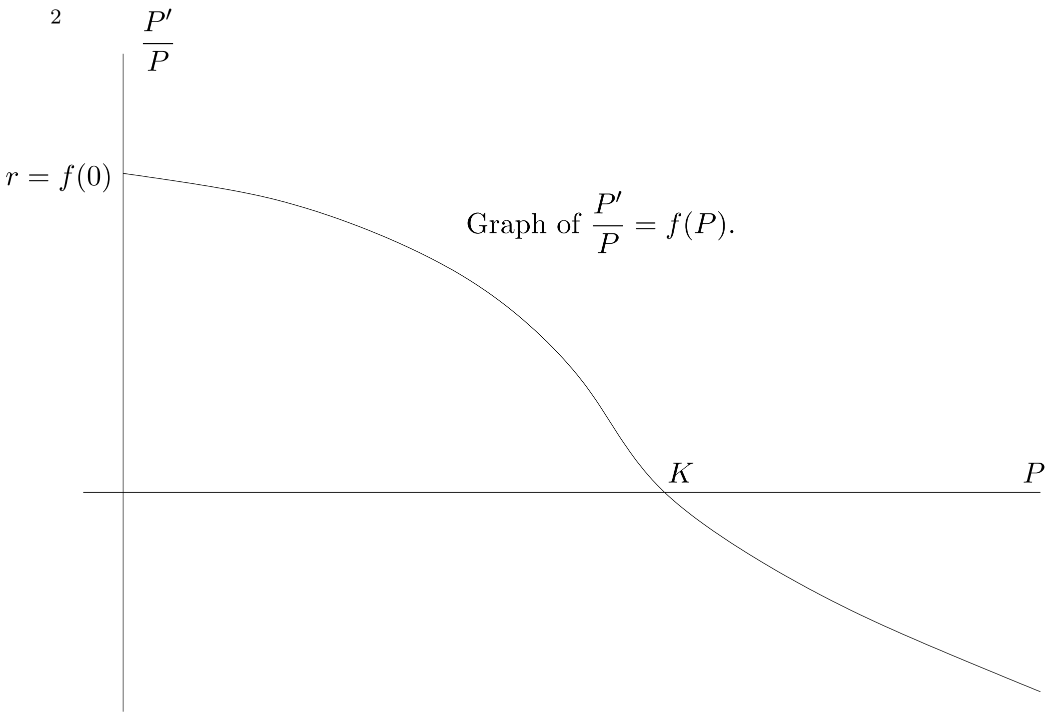
Reasonable assumptions on $f(P)$ are

- $f(P)$ gets smaller as P gets larger (that is f is decreasing) as when P gets larger the growth rate goes down due to crowding.
- $f(P)$ is positive for P close to $P = 0$. That if the population size is small, then the population is getting larger because over crowding not a factor for small populations.
- There is a population size $P = K$ where $f(K) = 0$. This is the ***carrying capacity*** and is the population size where the over crowding forces the population to stop growing.

In this model we let

$$r = f(0) = \text{unconstrained intrinsic growth.}$$

That is the graph will look something like the following:



So we will use for our basic model of population growth with limited resources the rate equation

$$\frac{P'}{P} = f(P).$$

Multiple by P to clear of fractions to get

$$P' = Pf(P).$$

Where, to summarize,

$$\begin{cases} f(P) > 0, & \text{for } 0 \leq P < K \\ f(K) = 0, & \\ f(P) < 0, & \text{for } K < P. \end{cases}$$

Problem 1. Show that for the rate equation

$$\frac{dP}{dt} = Pf(P)$$

with $f(P)$ as above that there are two equilibrium points. What are they? Also determine with is stable and which is unstable. □

In practice we usually use the simplest choice for $f(P)$, which is a stright line through the points $(0, r)$ and $(K, 0)$, that is

$$f(P) = r\left(1 - \frac{P}{K}\right).$$

The result is the ***logistic equation***

$$\frac{dP}{dt} = rP\left(1 - \frac{P}{K}\right).$$

Problem 2. For the logistic equation

$$\frac{dP}{dt} = .08\left(1 - \frac{P}{500}\right)$$

(a) Find the equilibrium points.

(b) Sketch a graph of the equilibrium solutions along with the solutions with $P(0) = 200$ and $P(0) = 600$.

(c) Which of the equilibrium points are stable?

(d) If $P(0) = 200$ estimate $P(100)$.

(e) If $P(0) = 700$ estimate $P(100)$.