Mathematics 172 Homework

In class today we looked at models of population growth where the population size the next year only depends on the size of the population size this year. That is if P_t is the size of the population in year t, then the population size the next year, that is P_{t+1} is given by

$$P_{t+1} = f(P_t)$$

for some function $f(P)$. A system like this, that is $P_{t+1} = f(P_t)$ is often called either a ***difference equation*** or a ***discrete dynamical system***.

As an example, consider

$$P_{t+1} = 2.1P_t - \frac{P_t^2}{100} \qquad \text{and} \qquad P_0 = 40.$$

Then

$$\begin{aligned} P_1 = 2.1P_0 - \frac{P_0^2}{100} &= 2.1(40) - \frac{(40)^2}{100} = 68.000 \\ P_2 = 2.1P_1 - \frac{P_1^2}{100} &= 2.1(68.000) - \frac{(68.000)^2}{100} = 96.560 \\ P_3 = 2.1P_2 - \frac{P_2^2}{100} &= 2.1(96.560) - \frac{(96.560)^2}{100} = 109.538 \end{aligned}$$

1. Show that for the difference equation

$$N_{t+1} = \frac{20N_t}{1 + .2N_t^2}$$

and $N_0 = 5$ that

$$\begin{aligned} N_0 &= 5.000 \\ N_1 &= 16.667 \\ N_2 &= 5.894 \\ N_3 &= 14.832 \\ N_4 &= 6.592 \end{aligned}$$

We also saw in class today that if we assume that a population grows with an intrinsic growth rate of r (when the population size is small) but there is a ***carrying capacity***, K, where for populations of size larger than K the per capita growth rate becomes negative, that a reasonable model for the growth of the population size is

$$P_{t+1} = P_t + rP_t\left(1 - \frac{P_t}{K}\right).$$

Assume the difference equation is

$$P_{t+1} = P_t + .3P_t\left(1 - \frac{P_t}{100}\right).$$

Assume that we know that in some year the population is $P_0 = 80$. Then the population the next year is

$$P_1 = P_0 + .3P_0\left(1 - \frac{P_0}{100}\right) = 80 + .3P_t\left(1 - \frac{80}{100}\right) = 84.8.$$

And the population the year after that is

$$P_1 = P_0 + .3P_1\left(1 - \frac{P_1}{100}\right) = 84.8 + .3P_t\left(1 - \frac{84.8}{100}\right) = 88.666880.$$

2. Show that for this difference equation we have

$$\begin{aligned} P_0 &= 80.000 \\ P_1 &= 84.800 \\ P_2 &= 88.667 \\ P_3 &= 91.681 \\ P_4 &= 93.969 \\ P_5 &= 95.670 \end{aligned}$$

3. Let $P_{t+1} = f(P_t)$ be a difference equation and let P_* be a number such that $f(P_*) = P_*$. Such points are called ***equilibrium*** points of the equation. Then show that if $P_0 = P_*$ that $P_t = P_*$ for all t. *Solution:* Here is the idea. $P_1 = f(P_0) = f(P_*) = P_*$. Therefore $P_1 = P_*$. Now $P_2 = f(P_1) = f(P_*) = P_*$. Now you use similar calculations to show that $P_3 = P_4 = P_5 = P_*$. This pattern continues for all t.
4. Find all the equilibrium points of

$$P_{t+1} = P_t + .3P_t\left(1 - \frac{P_t}{200}\right).$$

Solution: This is no more that a complicated way of asking us to find the solutions to the equation

$$P = P + .3P\left(1 - \frac{P}{200}\right).$$

Subtracting P from both sides gives

$$0 = .3P\left(1 - \frac{P}{200}\right).$$

and now a bit of algebra shows that the only equilibrium points are

$$P_* = 0 \qquad \text{and} \qquad P_* = 200$$