

## Math 172 Homework.

We have started talking about discrete dynamical systems which for us are numbers  $N_t$  where  $t = 0, 1, 2, 3, \dots$  and such that is a function  $f$  with

$$N_{t+1} = f(N_t).$$

If  $N_t$  is the population size in year  $t$ , then the function  $f$  lets us compute the population size the next year by the formula above.

In a discrete dynamical system  $N_{t+1} = f(N_t)$  the **equilibrium points** (also called a **stationary point** or a **rest point**),  $N_*$ , are the points solutions to  $f(N) = N$ . These are of interest because then if  $N_0 = N_*$ , then  $N_t = N_*$  for all  $t$ . That is the population stays the same from year to year.

The equilibrium point  $N_*$  is **stable** (or **attracting**) if when  $N_0$  starts close to  $N_*$ , then  $N_t$  gets closer and closer to  $N_*$ .

The equilibrium is **unstable** (or **repelling**) if when  $N_0$  is close to  $N_*$ , then  $N_t$  moves away from  $N_*$ .

We will see that there is an easy way to tell if  $N_*$  is stable. If  $|f'(N_*)| < 1$ , then  $N_*$  is a stable equilibrium point. If  $|f'(N_*)| > 1$  it is unstable.

**Problem.** In the special case of the discrete logistic equation with carrying capacity  $K$  and per capita grow rate  $r$ , that is

$$(1) \quad N_{t+1} = N_t + rN_t \left(1 - \frac{N_t}{K}\right)$$

That is in this case

$$f(N) = N + rN \left(1 - \frac{N}{K}\right)$$

(a) Solve the equation  $f(N) = N$  to see that there are two equilibrium points,  $N_* = 0$  and  $N_* = K$ .

(b) Compute  $f'(N)$  and show that

$$f'(0) = 1 + r$$

$$f'(K) = 1 - r.$$

(c) Use this to explain why the following hold

- The equilibrium at  $N = 0$  is always unstable.
- The equilibrium at  $N = K$  is always unstable is stable when  $0 < r < 2$  and unstable when  $r > 2$ .  $\square$

(d) Note that the right hand of (1) side of this equation also has a zero when  $N = \frac{1+r}{r}K$ . Does this have any biological meaning. *Answer:* At least to the best of my knowledge the answer is no.

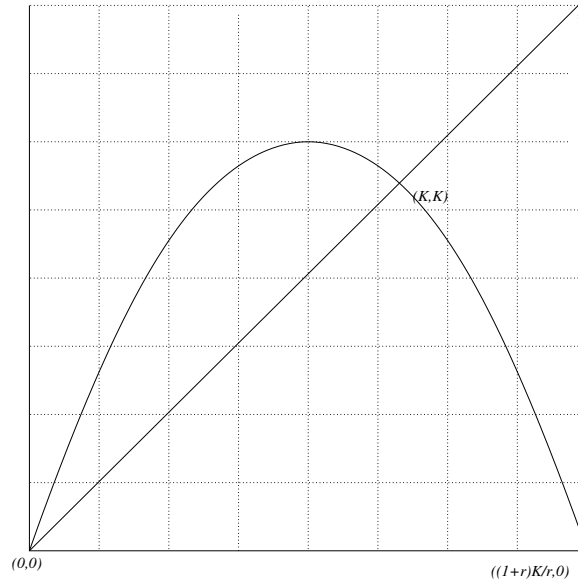


FIGURE 1. The graph of  $N_{t+1} = N_t + rN_t \left(1 - \frac{N_t}{K}\right)$  as a function of  $N_t$ , showing the two equilibrium points  $(0, 0)$  and  $(K, K)$ . Also showing that the zeros of the right hand side of (1) are  $N = 0$  and  $N = \frac{1+r}{r}K$ .

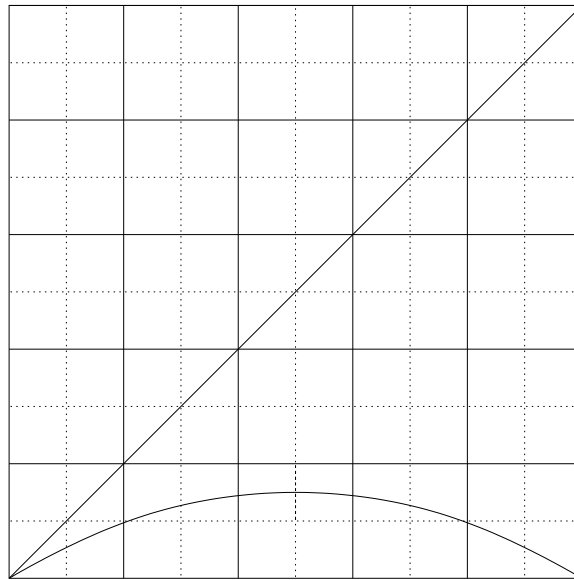


FIGURE 2. In this system there is only one equilibrium point, the one at  $N = 0$ . Is it stable or unstable? Draw some cobwebs to decide.

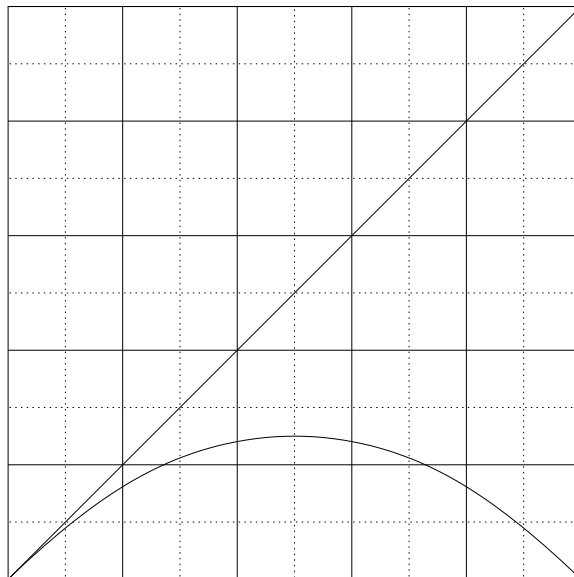


FIGURE 3. Again there is only one equilibrium point, the one at  $N = 0$ . Is it stable or unstable? Draw some cobwebs to decide.

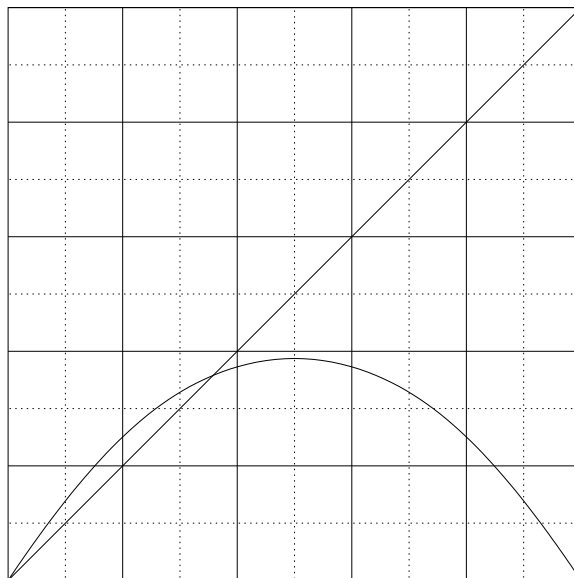


FIGURE 4. Here there are two equilibrium points. Are they stable or unstable?

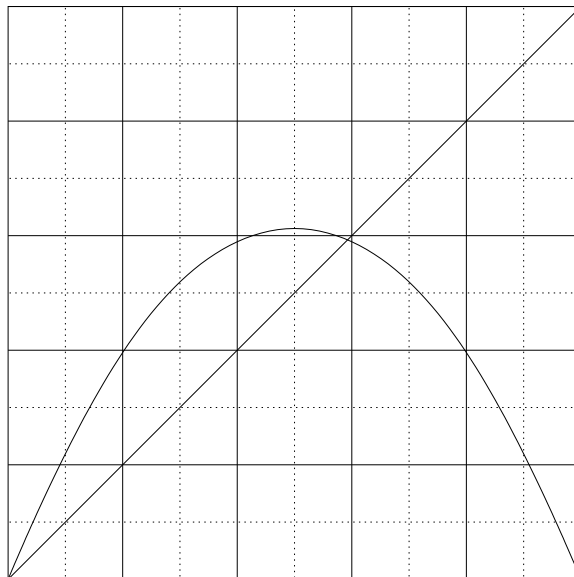


FIGURE 5. Again decide if the two equilibrium points are stable or unstable.

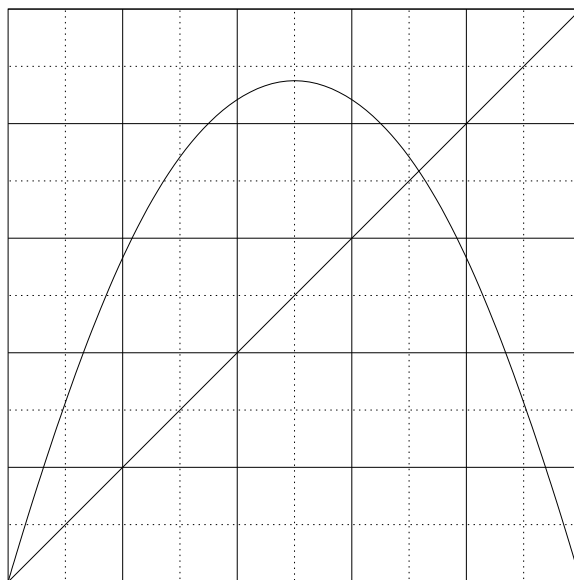


FIGURE 6. Are the two equilibrium points are stable or unstable.

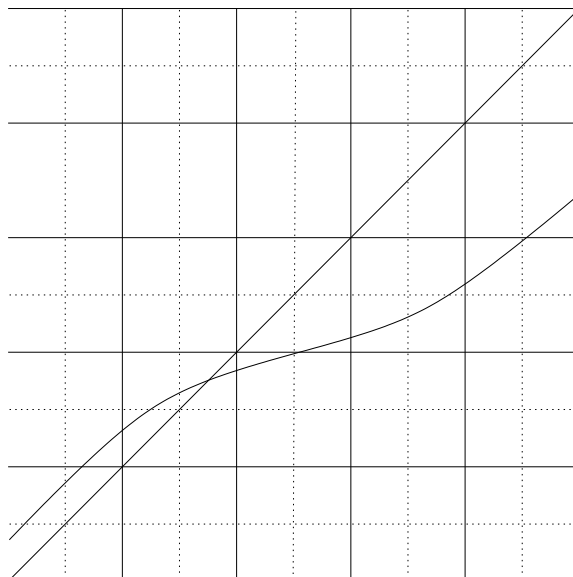


FIGURE 7. Is the equilibrium point are stable or unstable?

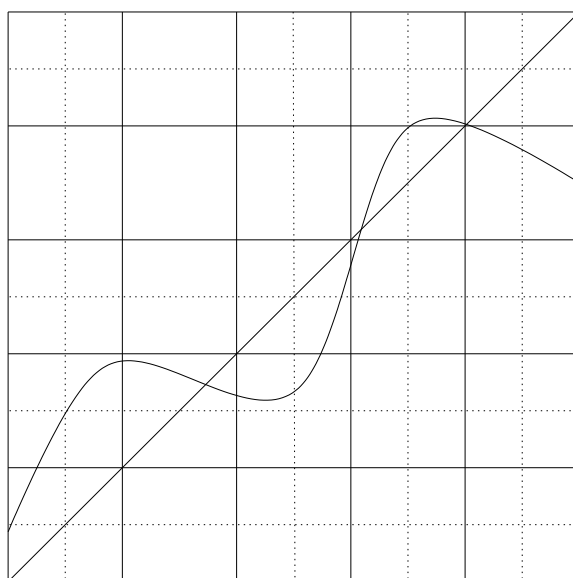


FIGURE 8. Here there are three equilibrium points. Which are stable and which unstable?

Answers:

Figure 2: 0 is stable.

Figure 3: 0 is stable.

Figure 4: 0 is unstable and the other rest point is stable.

Figure 5: 0 is unstable and the other rest point is stable.

Figure 6: Both points are unstable.

Figure 7: The point is stable.

Figure 8: The middle rest point is unstable, the other two are stable.