Mathematics 739 Homework 3: Differential forms.

On $\mathbb{R}^n$ a zero form is just a smooth function $f\colon \mathbb{R}^n \to \mathbb{R}$. A one form is an expression

$$\alpha = \sum_{j=1}^{n} a_j \, dx^j$$

where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$ and the a_j's are smooth functions. We also view each dx^j as a linear functional on $\mathbb{R}^n$ by letting

$$dx^j \left(\sum_{k=1}^{n} v^k \frac{\partial}{\partial x^k} \right) = v^j$$

where $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ is the standard basis of $\mathbb{R}^n$. Here we are identifying vectors with point deprivations. That is if $p \in \mathbb{R}^n$ and v is a vector at p (i.e. $v \in TM_p$) then we can also view v as the directional derivative in the direction of v. That is if f is a smooth real valued function, then

$$v(f) = \frac{d}{dt} f(p + tv)\Big|_{t=0}.$$

This operator satisfies that $f \mapsto v(f)$ is linear over $\mathbb{R}$ and than

$$v(fg) = f(p)v(g) + v(f)g(p).$$

Proposition 1. *If V is an operator on smooth real valued functions on $\mathbb{R}^n$ such that V is linear over $\mathbb{R}$ and for some point $p \in \mathbb{R}^n$*

$$V(fg) = V(f)g(p) + f(p)V(g)$$

then there is a vector v at p such that V is given by

$$V(f) = \frac{d}{dt} f(p + tv)\Big|_{t=0}.$$

Thus V is naturally identified with a vector to $\mathbb{R}^n$ at the point p.

Problem 1. Prove this. *Hint:* First show $V(c) = 0$ for any constant c. Then show if h_1 and h_2 are smooth functions with $h_1(p) = h_2(p) = 0$, then for any smooth function g that $V(h_1 h_2 g) = 0$. Now use some form or anther of Taylor's theorem to write the smooth function f as

$$f = f(p) + \sum_{j=1}^{n} a_j (x^j - p^j) + \sum_{j,k=1}^{n} (x^j - p^j)(x^k - p^k) g_{jk}$$

where the a_j's are constants and the $g_{jk}s$'s are smooth functions. Put this all together to conclude

$$V(f) = \sum_{j=1}^{n} a_j V(x^j) = \frac{d}{dt} f(p + tv)\Big|_{t=0}$$

where v is the vector $v = \sum_{j=1}^{n} a^j \frac{\partial}{\partial x^j}$. $\square$

One reason for viewing vectors this way is that this definition is easy to generalize to manifolds. Let M be a smooth manifold and, $C^\infty(M)$ the algebra of smooth real valued functions on M and $p \in M$ a point. Then we can define a ***point derivation*** at p to be a map $f \mapsto V(f)$ which is linear over $\mathbb{R}$ and such that for $f, g \in \mathbb{C}^\infty(M)$

$$V(fg) = V(f)g(p) + f(p)V(g).$$

Then the set of all such point derivations at p form the tangent space, TM_p, to M at p. To make this definition a bit more geometric let $c\colon (-\delta, \delta)$ be a smooth curve with $c(0) = p$. Then

$$V(f) = \frac{d}{dt} f(c(t))\bigg|_{t=0}$$

is a point derivation at p which we denote, naturally enough, at $c'(0)$. It is the tangent vector to c at $t = 0$. An easy extension of Proposition 1 shows that every $v \in TM_p$ can be realized as the tangent vector to a curve through p.

If $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n$ is anther coordinate system on $\mathbb{R}^n$ and let the one form α be given in the two coordinate systems by

$$\alpha = \sum_{j=1}^n a_j\, dx^j = \sum_{j=1}^n \tilde{a}_j\, d\tilde{x}^j.$$

Then

$$\tilde{a}_j = \sum_{k=1}^n \frac{\partial x^k}{\partial \tilde{x}^j} a_k.$$

This is often written as

$$\tilde{a}_j = \frac{\partial x^k}{\partial \tilde{x}^j} a_k$$

with the convention that we sum over any repeated index.[1]

Problem 2. Prove this transformation rule. Also show that if a vector field is given in the two coordinates systems as

$$\sum_{j=1}^n a^j \frac{\partial}{\partial x^j} = \sum_{j=1}^n \tilde{a}^j \frac{\partial}{\partial \tilde{x}^j}$$

then

$$\tilde{a}^j = \sum_{k=1}^n \frac{\partial \tilde{x}^j}{\partial x^k} a^k = \frac{\partial \tilde{x}^j}{\partial x^k} a^k.$$

$\square$

[1]This convention seems to have been introduced by Einstein in his paper *Die Grundlage der allgemeinen Relativitätstheorie* in Annalen der Physik in 1916. This is why it is often called the Einstein summation convention.

If $f\colon \mathbb{R}^n \to \mathbb{R}$ is a smooth function we define its ***differential*** (also called its ***exterior derivative***) by

$$df = \sum_{=1}^{n} \frac{\partial f}{\partial x^j}\, dx^j.$$

The chain rule shows that this is the linear functional defined on vectors by

$$df_p(v) = \frac{d}{dt} f(p+tv)\bigg|_{t=0}.$$

Problem 3. Show that the definition of df is independent of the coordinate system used to define it. □

If $1lek \leqslant n$ a smooth k-form is sum of the form

$$\alpha = \sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant n} a_{j_1 j_2\cdots j_k}\, dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_k}$$

where each of the $a_{j_1 j_2\cdots j_k}$ are smooth functions. The wedge product $\wedge$ is so that

$$dx^j \wedge dx^k = -dx^k \wedge dx^j$$

which implies that for any j

$$dx^j \wedge dx^j = 0.$$

The products $dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_k}$ can be view as k-linear alternating functions as follows. For $k=2$

$$dx^{j_1} \wedge dx^{j_2}(u,v) = dx^{j_1}(u)dx^{j_2}(v) - dx^{j_1}(v)dx^{j_2}(u) = \det \begin{bmatrix} dx^{j_1}(u) & dx^{j_2}(v) \\ dx^{j_1}(v) & dx^{j_2}(u) \end{bmatrix}$$

and in general

$$dx^{j_1} \wedge dx^{j_2} \wedge \cdots \wedge dx^{j_k}(v_1, v_2, \ldots, v_k) = \det\left(\left[dx^{j_s}(v_t)\right]_{s,t=1}^{k}\right).$$

In writing differential forms it is useful to use the multi-index notation. Let $J = (j_1, j_2, \ldots, j_k)$ then set

$$dx^J = dx^{j_1} \wedge dx^{j_2} \wedge \cdots dx^{j_k}.$$

Problem 4. With this notation

(a) If J has a repeated index, then $dx^J = 0$.
(b) If the elements of J' are a permutation of the elements of J, say $J' = (j_{\sigma(1)}, j_{\sigma(2)}, \ldots, j_{\sigma(k)})$ with σ a permutation of $\{1, 2, \ldots, k\}$, then $dx^{J'} = \operatorname{sign}(\sigma) dx^J$.
(c) If J and L have an element in common, then $dx^J \wedge dx^L = 0$.
(d) If J and L have no element in common and J has degree k and L has degree ℓ, then $dx^L \wedge dx^J = (-1)^{k\ell} dx^J \wedge dx^L$.

We can now write a k form α as

$$\alpha = \sum_J a_J \, dx^J$$

where, depending on which is more useful in a given context, the sum is either over all length k multi-indices or over all increasing multi-indices.

If α and β are forms, say

$$\alpha = \sum_J a_J \, dx^J, \qquad \beta = \sum_L b_L \, dx^L$$

then the ***wedge product*** (also called the ***exterior product***) of these is

$$\alpha \wedge \beta = \sum_{J,L} a_J b_L \, dx^J \wedge dx^L.$$

Problem 5. Show this product is associative and its definition is independent of the coordinate system used. $\square$

Problem 6. Let α be a k-form and β a ℓ-form.

(a) Show that $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$.
(b) Show that if k is odd, then $\alpha \wedge \alpha = 0$.
(c) Let $\omega = dx^1 \wedge dx^2 + dx^2 \wedge dx^4$. Show $\omega \wedge \omega = 2\, dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \neq 0$. Thus it is not true $\alpha \wedge \alpha = 0$ for all forms. $\square$

We can now extend the definition of the differential, df, of a smooth function to general forms. Let

$$\alpha = \sum_J a_J \, dx^J.$$

Then its ***exterior derivative*** is

$$d\alpha = \sum_J da_J \wedge dx^J.$$

Proposition 2. *This definition is independent of the coordinate system used to define it. Also*

(a) For any form α

$$dd\alpha = 0.$$

(b) If α is a k form and β is a ℓ form

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^k \alpha \wedge (d\beta).$$

A k-form α is called ***closed*** if $d\alpha = 0$ and ***exact*** if $\alpha = d\beta$ for some $(k-1)$-form β.

Theorem 3 (Poincaré lemma)**.** *Let α be a smooth form defined on a contractable open subset U of $\mathbb{R}^n$ with $\deg(\alpha) \geqslant 1$. If $d\alpha = 0$, then there is a form β with*

$$d\beta = \alpha.$$

That is on a contractable open set closed forms are exact.

Problem 7. Prove the following special case of the Poincaré lemma:[2] Let $U := \bigtimes_{j=1}^{n}(a_j, b_j)$ be an open rectangular parallelepiped in $\mathbb{R}^n$ and let α be a closed k form on U. Then there is a $(k-1)$-form β with $d\beta = \alpha$. *Hint:* We use induction on $n + \deg(\alpha)$. Here we show the induction step in reducing the case$n = 4$ and $k = 2$ to a lower dimensional case. You should be able to adapt thus to the general case. Let x, y, z, w be coordinates on $\mathbb{R}^4$. Let α be a closed two form and write it as

$$\alpha = \alpha_0 + P\, dx \wedge dw + Q\, dy \wedge dw + R\, dz \wedge dw.$$

where α does not have any factors of dw. Show that there are functions p, q, and r on U such that

$$\frac{\partial p}{\partial w} = P, \quad \frac{\partial q}{\partial w} = Q, \quad \frac{\partial r}{\partial w} = R.$$

Let

$$\beta_1 = p\, dx + q\, dy + r\, dz$$

and set

$$\alpha_1 = \alpha + d\beta_1.$$

Then

$$\begin{aligned}\alpha_1 = \alpha_0 &+ P\, dx \wedge dw + Q\, dy \wedge dw + R\, dz \wedge dw \\ &- \frac{\partial p}{\partial w} dx \wedge dw - \frac{\partial q}{\partial w} dy \wedge dw - \frac{\partial r}{\partial w} dz \wedge dw \\ &+ \text{terms from } d\beta_1 \text{ that have no factor of } dw.\end{aligned}$$

Therefore α_1 satisfies $d\alpha_1 = 0$ and α_1 has no factors of dw. Write

$$\alpha_1 = A\, dx \wedge dy + B\, dx \wedge dz + C\, dy \wedge dz.$$

Then $d\alpha_1 = 0$ implies that the coefficients of $dx \wedge dy \wedge dw$, $dy \wedge dz \wedge dw$, $dx \wedge dz \wedge dw$ in the expansion of $d\alpha_1$ vanish. That is

$$\frac{\partial A}{\partial w} = \frac{\partial B}{\partial w} = \frac{\partial C}{\partial w} = 0.$$

Since U is a convex domain these equation imply that A, B, and C are independent of w. That is A, B, and C are functions of (x, y, z). So

$$\alpha_1 = A(x, y, z)\, dx \wedge dy + B(x, y, z)\, dx \wedge dz + C(x, y, z)\, dy \wedge dz$$

is a form on a lower dimensional rectangular parallelepiped. Whence by the induction hypothesis there is a form β_2 with

$$d\beta_2 = d\alpha_1.$$

Putting together with $\alpha_1 = \alpha + d\beta_1$ gives

$$\alpha = \alpha_1 - d\beta_1 = d\beta_2 - d\beta_1 = d\beta$$

where $\beta = \beta_2 - \beta_1$. This completes the induction and the proof. $\square$

[2]This proof is based on some notes from the web site of James Marrow: `https://sites.math.washington.edu/~morrow/335_12/335.html` and is based on a proof in Walter Rudin's *Principles of Mathematical Analysis.*

So we can summarize the above as "exactly forms are closed" and conversely "a closed form on a contractable set is exact". It is not true that all closed forms are exact. Probably the best known example is

$$\alpha = d\arctan(y/x) = \frac{x\,dy - y\,dx}{x^2 + y^2}$$

which is closed on $\mathbb{R}^2 \backslash \{(0,0)\}$, but is not exact.

Let M be a smooth manifold with $\dim(M) = n$. For each k with $0 \leqslant k \leqslant n$ let

$$A^k(M) = \text{vector space of all smooth } k\text{-forms on } M.$$

Also let

$$Z^p(M) = \{\alpha \in A^k(M) : d\alpha = 0\}$$

be the vector space of all closed forms on M. Set

$$H^k_{\mathrm{dR}}(M) = Z^k(M)/dA^{k-1}(M)$$

with the convention that $A^{-1}(M) = \{0\}$. These are the ***de Rham cohomology groups*** of M.

Proposition 4. *The set*

$$H^*_{\mathrm{dR}}(M) = \bigoplus_{k=0}^{\dim(M)} H^k_{\mathrm{dR}}(M)$$

is an algebra where the multiplication is

$$[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$$

where $[\omega]$ *is the cohomology class of the closed form* ω*.*

Problem 8. Prove this. *Hint:* Once you have shown the product is well defined everything else falls out easily. $\square$

As a trivial example:

Proposition 5. *If* U *is a contractable open set in* $\mathbb{R}^n$*, then* $H^*_{\mathrm{dR}}(U) = \{0\}$

$$H^*_{\mathrm{dR}}(U) = \begin{cases} \mathbb{R}, & k = 0; \\ 0, & k \neq 0. \end{cases}$$

Proof. For $k > 0$, that $H^k_{\mathrm{dR}}(U) = 0$ is nothing more than a restatement of the Poincaré lemma. For $k = 0$, if α is a closed 0-form, then α is just a smooth function $\alpha\colon U \to \mathbb{R}$. That α is closed, that is $d\alpha = 0$, in this case implies that α is locally constant and as U is connected this implies that α is constant. But there are no forms of degree -1 so the only exact form is $\alpha = 0$. Thus $H^0_{\mathrm{dR}}(U) = \mathbb{R}/\{0\} = \mathbb{R}$. $\square$

To give a less trivial example let $T^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus. Then the forms dx^1, dx^2,…,dx^n are translation invariant and therefore make sense on T^n. The algebra $H^*_{\mathrm{dR}}(T^n)$ is the algebra generated by the cohomology classes $[dx^1]$, $[dx^2]$,…, $[dx^n]$.

Problem 9. On T^2 show that all of the cohomology classes $[dx]$, $[dy]$ and $[dx \wedge dy]$ are nonzero. $\square$

Anther important property of differential forms is how they transform under smooth maps. To start let $f\colon M \to N$ be a smooth map between smooth manifolds, and let $a\colon N \to \mathbb{R}$ be a smooth function. That is a is a 0-form. Then we pull back a to M in the usual way:

$$f^*a := f \circ a.$$

Let α be a one form on N. Than for each $y \in N$ we have that $\alpha_y\colon TN_y \to \mathbb{R}$ is a linear functional. We can then define

$$(f^*\alpha)_x(v) = \alpha_{f(x)}(Df_x(v))$$

where v is a vector in TM_x and Df_x is the derivative of f at x. Then $(f^*\alpha)_x$ is a linear functional on TM_x and thus $f^*\alpha$ is a 1-form on M. Likewise if α is a 2-form on N, then for each $y \in N$ we have that $\alpha_y\colon TN_y \times TN_y \to \mathbb{R}$ is an alternating bilinear function. Then we $f^*\alpha$ is the 2-from on M given by

$$(f^*\alpha)_x(u,v) = \alpha_f(x)(Df_x(u), Df_x(v)).$$

This definition clearly generalizes to to k-forms by viewing them as alternating k-linear functions on tangent spaces. We have the basic transformation rule $(f \circ g)^* = g^* \circ f^*$ which we now state in a highbrow may.

Proposition 6. *The map $M \mapsto A^k(M)$ that sends a smooth manifold M to the module (over $C^\infty(M)$) is contravariant functor from the category of smooth manifolds and smooth maps, to the category of rings and modules.*

Problem 10. Make this precise and prove it. $\square$

The properties of the next proposition are also very important, but less obvious than the property of the last proposition.

Proposition 7. *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Let α be a k-form on N. Then*

$$d(f^*\alpha) = f^*d\alpha$$

and if β is a ℓ form then

$$f^*(\alpha \wedge \beta) = (f^*\alpha) \wedge (f^*\beta).$$

Problem 11. Chase through the definitions and prove this. Or more realistically go through your Math 738 notes and look at the proof there. Or find a readable book on differential geometry and see how it is proven there. $\square$

Proposition 8. *Let $f\colon M \to N$ be a smooth map between smooth manifolds. Then $f^*\colon H^*_{\mathrm{dR}}(N) \to H^*_{\mathrm{dR}}(M)$ given by*

$$f^*[\alpha] = [f^*\alpha]$$

is well defined and gives is a contravariant functor from the category of smooth manifolds and maps to the category of graded algebras.

Problem 12. Prove this (after looking up the definition of "graded algebra" if you don't remember what one is). □

Example 9. Let M_1 and M_2 be connected smooth manifolds. Let $M = M_1 \times M_2$. Then we have the projections $p_j \colon M \to M_j$. Also let $x_j \in M_j$ and we have inclusions $\iota \to M$ by

$$\iota(x) = (x, x_2), \qquad \iota(x) = (x_1, x).$$

Then $p_j \circ \iota_j = I_{M_j}$. Therefore for the induced maps on de Rham cohomology we have

$$\iota_j^* \circ p_j^* = I_{H^*_{\mathrm{dR}}(M_j)}.$$

This tells us that $\iota_j^* \colon H^*_{\mathrm{dR}}(M) \to H^*_{\mathrm{dR}}(M_j)$ is surjective and $p_j^* \colon H^*_{\mathrm{dR}}(M_j) \to H^*_{\mathrm{dR}}(M)$ is injective.

We next define the integration of differential forms. If ω is an n-form on $\mathbb{R}^n$, then it is of the form

$$\omega = f(x)\, dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n.$$

Assume that f has compact support, that is $\operatorname{spt}(f) := \overline{\{x : f(x) \neq 0\}}$ is compact, then we define

$$\int_{\mathbb{R}^n} \omega := \int f\, dx^1 dx^2 \cdots dx^n$$

where $dx^1 dx^2 \cdots dx^n$ is the usual Lebesgue measure defined by this coordinate system. Now here is the magic part of the of the definition of the wedge product. Let $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ be anther coordinate system on $\mathbb{R}^n$. Then in this coordinate system the form ω is given by

$$\omega = \tilde{f}\, d\tilde{x}^1 \wedge d\tilde{x}^2 \wedge \cdots \wedge dx^n$$

where

$$\tilde{f} = \det \left[\frac{\partial x^i}{\partial \tilde{x}^j} \right] f.$$

Proposition 10. *If* $\det \left[\frac{\partial x^i}{\partial \tilde{x}^j} \right] > 0$ *then*

$$\int_{\mathbb{R}^n} \tilde{f}\, d\tilde{x}^1 d\tilde{x}^2 \cdots d\tilde{x}^n = \int_{\mathbb{R}^n} f\, dx^1 dx^2 \cdots dx^n.$$

Problem 13. Use the change of variable formula from advanced calculus to prove this. □

Call a coordinate system $\tilde{x}^1, \tilde{x}^2, \dots, \tilde{x}^n$ a ***positive coordinate*** system if $\det \left[\frac{\partial x^i}{\partial \tilde{x}^j} \right] > 0$. The last proposition shows that for a compactly support n form, ω, that $\int_{\mathbb{R}^n} \omega = \int \tilde{f}\, d\tilde{x}^1 d\tilde{x}^2 \cdots d\tilde{x}^n$ is defined is defined independently of the choice of the coordinate system.

This gives a first step toward integrating forms on manifolds. Recall that a smooth manifold M is ***orientable*** if it has an atlas $\{(U_\alpha, \psi_\alpha)\}_{\alpha \in A}$ where all coordinate are positively related on their overlaps. That if for each α, β the map $\psi_\alpha \circ \psi_\beta^{-1}\big|_{\psi\beta[U_{\alpha\beta}]} \colon \psi_\beta[U_{\alpha\beta}] \to \psi_\alpha[U_{\alpha\beta}]$ (where $U_{\alpha\beta} = U_\alpha \cap U_\beta$) has positive Jacobian.

An ***orientation*** on a smooth manifold, M, is a choice of a positive atlas on M. Let ω be a compactly supported n-form on the oriented n-dimensional manifold M. Assume there is a positive coordinate system $x^1, x^2, \ldots, x^n$ with

$$\operatorname{spt}(\omega) \subseteq \text{domain of } x^1, x^2, \ldots, x^n.$$

Then in this coordinate system ω will have the form

$$\omega = f\, dx^1 \wedge dx^2 \cdots dx^n$$

and we can define

$$\omega = \int f\, dx^2 dx^2 \cdots dx^n.$$

Then if $\tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n$ is anther positive coordinate system with

$$\operatorname{spt}(\omega) \subseteq \text{domain of } \tilde{x}^1, \tilde{x}^2, \ldots, \tilde{x}^n.$$

and $\omega = \tilde(f) d\tilde{x}^1 \wedge dx^2 \wedge \cdots \wedge dx^n$ then Proposition 10 tells use this gives the same result as for our original coordinate system.

Therefore we have defined $\int_M \omega$ for n-forms, ω, on oriented n-dimensional manifolds when ω has small support. To globalise this definition we use partitions of unit.

Definition 11. Let M be a smooth manifold and $\mathcal{U}$ an open cover of M. Then a ***partition of unity*** subordinate to $\mathcal{U}$ is a collection $\{\phi_\alpha\}_{\alpha\in A}$ of smooth nonnegative real valued functions such that

(a) $\sum_{\alpha\in A} \phi_\alpha(x) = 1$ for all $x \in M$,
(b) For each $\alpha \in A$ there is $U \in \mathcal{U}$ with $\operatorname{spt}(\phi_\alpha) \subseteq U$.
(c) the sum $\sum_{\alpha\in A} \phi_\alpha$ is locally finite in the sense that each $x \in M$ has a neighborhood V such that the set $\{\alpha \in A : \operatorname{spt}(\phi_\alpha) \cap V\}$ is finite.

Theorem 12. *If M is a smooth manifold with a separable topology, then every open cover of M has a partition of unity subordinate to it.* □

Now let ω be a compactly supported n-form on the n-dimensional smooth manifold M. Let $\mathcal{U}$ be a collection of the domains of positive coordinate neighborhoods that cover M. Let $\{\phi_\alpha\}_{\alpha\in A}$ be a partition of unit subordinate to this open cover. Then we the integral of ω by

$$\int_M \omega = \sum_{\alpha\in A} \int_M \phi_\alpha \omega.$$

As each $\phi_\alpha\omega$ has support contained in a positive coordinate neighborhood, the integral $\int_M \phi_\alpha\omega$ is defined as above. And that ω is compactly support and the sum for the partition of unity is locally finite implies that only a finite number of the terms in the sum $\sum_{\alpha\in A} \int_M \phi_\alpha\omega$ only has finitely many nonzero terms so convergence is not a problem. If $\{\rho_\beta\}_{\beta\in B}$ is anther partition subordinate to anther cover by positive coordinate systems we have

$$\begin{aligned}
\sum_{\alpha\in A}\int_M \phi_\alpha\omega &= \sum_{\alpha\in A}\int_M 1\phi_\alpha\omega \\
&= \sum_{\alpha\in A}\int_M \left(\sum_{\beta\in B}\rho_\beta\right)\phi_\alpha\omega \\
&= \int_M \left(\sum_{\beta\in B}\rho_\beta\right)\left(\sum_{\alpha\in A}\phi_\alpha\right)\omega \\
&= \int_M \left(\sum_{\beta\in B}\rho_\beta\right)1\,\omega \\
&= \sum_{\beta\in B}\int_M \rho_\beta\omega
\end{aligned}$$

which shows that this definition is independent of the partition of unity used.

Lemma 13. *Let σ be an $(n-1)$-form on $\mathbb{R}^n$. Then*

$$\int_{\mathbb{R}^n} d\omega = 0.$$

Problem 14. Prove this. *Hint:* Here is the proof when $n = 3$. You should not have much trouble generalizing it. Let

$$\sigma = A dx \wedge dy + B dx \wedge dz + C dy \wedge dz.$$

Then

$$d\sigma = \left(\frac{\partial A}{\partial z} - \frac{\partial B}{\partial y} + \frac{\partial C}{\partial x}\right) dx \wedge dy \wedge dz.$$

We now take the first of these terms.

$$\begin{aligned}
\int_{\mathbb{R}^3} \frac{\partial A}{\partial z}\, dx \wedge dy \wedge dz &= \int_{\mathbb{R}^3} \frac{\partial A}{\partial z}\, dxdydz \\
&= \int_{\mathbb{R}^2}\left(\int_{-\infty}^{\infty}\frac{\partial A}{\partial z}(x,y,z)\,dz\right)dxdy \\
&= 0
\end{aligned}$$

where $\int_{-\infty}^{\infty}\frac{\partial A}{\partial z}(x,y,z)\,dz = A(x,y,\infty) - A(x,y,-\infty) = 0$ because A is compactly supported. The proofs for the other two terms is identical. $\square$

Theorem 14 (Stokes' Theorem)**.** *If σ is a compactly supported $(n-1)$-form on an oriented n-dimensional manifold M, then*

$$\int_M d\sigma = 0.$$

Proof. If σ is supported in a positive coordinate system, then the proof that $\int_M d\sigma = 0$ reduces to that of Lemma 13. So we can choose a partition of unity $\{\phi_\alpha\}_{\alpha \in A}$ such that for each α

$$\int_M d(\phi_\alpha \sigma) = 0.$$

But then

$$\begin{aligned}\int_M d\sigma &= \int_M d(1\sigma) \\ &= \int_M d\Big(\Big(\sum_{\alpha \in A} \phi_\alpha\Big)\sigma\Big) \\ &= \sum_{\alpha \in A} \int_M d(\phi_\alpha \sigma) \\ &= 0\end{aligned}$$

□

Here is anther result that is easily proven using partitions of unity.

Proposition 15. *A smooth manifold M is orientable if and only if there is a nowhere vanishing n-form on M.*

Problem 15. Prove this. *Hint:* The easy direction is that if M has a nowhere vanishing n-form ω, then for any local coordinate system $x^1, x^2, \dots, x^n$ on a connected open set we have $dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n = f\omega$ where f is a nonvanishing smooth function. Thus f is either always positive, or always negative. Call the coordinate system positive if f is positive. Show the collection of coordinate systems where the function f is positive is an oriented atlas for M.

Conversely if M is covered by a collection of charts $(U, x^1, \dots, x^n)$ where all overlaps are positive, then on to such charts $(U, x^1, \dots, x^n)$ and $(\tilde{U}, \tilde{x}^1, \dots, \tilde{x}^n)$ on the overlap $U \cap \tilde{U}$ we have that $dx^1 \wedge \cdots \wedge dx^n$ and $d\tilde{x}^1 \wedge \cdots \wedge d\tilde{x}^n$ are pointwise positive multiples of each other. Now piece these n-forms defined on the coordinate neighborhoods together by using a partition of unity. □

Theorem 16 (Mayer-Vietoris Sequence)**.** *Let M smooth manifold and assume that $M = U \cup V$ where U and V are open subsets of M. Then the following sequence of chain complexes is exact.*

$$0 \longrightarrow A^*(M) \xrightarrow{\ r\ } A^*(U) \oplus A^*(V) \xrightarrow{\ s\ } A^*(U \cap V) \longrightarrow 0$$

where

$$r(\alpha) = (\alpha\big|_U, \alpha\big|_V) \quad \textit{and} \quad s(\alpha, \beta) = \alpha\big|_U - \beta\big|_V.$$

Proof. That r is injective is clear and it is not hard to see $\ker(s) = \mathrm{Image}(r)$. So we only need to show s is surjective. Let $\gamma \in A^*(U \cap V)$. Let $\{\rho_V, \rho_V\}$ be a partition of unity with $\mathrm{spt}(\rho_U) \subseteq U$ and $\mathrm{spt}(\rho_V) \subseteq V$. Then

$$s(\rho_U \gamma\big|_U, -\rho_V \gamma\big|_V) = \gamma.$$

This shows s is surjective and completes the proof. □

As we have seen in the algebraic topology class, a short exact sequences of chain complexes leads to a long exact sequence in cohomology:

$$\to H^{k-1}_{\mathrm{dR}}(U\cap V)\to H^k_{\mathrm{dR}}(M)\to H^k_{\mathrm{dR}}(U)\oplus H^K_{\mathrm{dR}}(V)\to H^k_{\mathrm{dR}}(U\cap V)\to H^{k+1}(M)\to$$

And we know that on contractable open sets that $H^*_{\mathrm{dR}}(U)=\mathbb{R}[1]$ (this is $H^*_{\mathrm{dR}}(M)$ is just the constant multiples of the class of the constant function 1). Therefore we can do calculations similar to the ones we did in algebraic topology to get results such as

$$H^k_{\mathrm{dR}}(S^n)=\begin{cases}\mathbb{R}, & k=0,n;\\ 0, & \text{otherwise.}\end{cases}$$

where $n\geqslant 1$. On the torus $T^n=\mathbb{R}^n/\mathbb{Z}^n$ we have

$$\dim H^*_{\mathrm{dR}}(T^n)=\binom{n}{k}$$

for $0\leqslant k\leqslant n$. And if M_g is the compact oriented surface of genus g, then

$$H^k_{\mathrm{dR}}(M_g)=\begin{cases}\mathbb{R}, & k=0,2;\\ \mathbb{R}^{2g}, & k=1.\end{cases}$$

Or we can go about this is a more highbrow manner by using sheaves. Recall that a sheaf, $\mathcal{S}$, on M is an assignment of a group (or ring or vector space or whatever is appropriate in context[3]), $\mathcal{S}(U)$ to each open subset U of M such that this assignment has some nice properties (which I am not going to rewrite here). Give any sheaf $\mathcal{S}$ we can construct cohomology groups $H^k(M,\mathcal{S})$ by a variant of the Čech construction by taking open covers and refining them. For use the important point is that given a short exact sequence

$$0\to\mathcal{A}\to\mathcal{B}\to\mathcal{C}\to 0$$

of sheaves on M that this leads to a long exact sequence in cohomology:

$$\begin{aligned}0&\to H^0(M,\mathcal{A})\to H^0(M,\mathcal{B})\to H^0(M,\mathcal{C})\\ &\to H^1(M,\mathcal{A})\to H^1(M,\mathcal{B})\to H^1(M,\mathcal{C})\\ &\to H^2(M,\mathcal{A})\to H^2(M,\mathcal{B})\to H^2(M,\mathcal{C})\\ &\qquad\qquad\vdots\\ &\to H^k(M,\mathcal{A})\to H^k(M,\mathcal{B})\to H^k(M,\mathcal{C})\\ &\to H^{k+1}(M,\mathcal{A})\to H^{k+1}(M,\mathcal{B})\to H^{k+1}(M,\mathcal{C})\\ &\qquad\qquad\vdots\end{aligned}$$

[3]In general for the construction here to work it is enough for the sheaf to take values in an Abelian category.

We recall a bit about the Čech construction. Let $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$ be a locally finite open cover of M. For each $p \geqslant 0$ and a sheaf $\mathcal{S}$ let

$$C^p(\mathcal{U},\mathcal{S}) = \prod_{(\alpha_0,\dots,\alpha_p)\in A^{p+1}} \mathcal{S}(U_{\alpha_0,\dots,\alpha_p})$$

where

$$U_{\alpha_0,\dots,\alpha_p} = U_{\alpha_0} \cap U_{\alpha_1} \cap \cdots \cap U_{\alpha_p}.$$

The ***coboundary operator***

$$\delta\colon C^p(\mathcal{U},\mathcal{S}) \to C^{p+1}(\mathcal{U},\mathcal{S})$$

is given by

$$(\delta s)_{\alpha_0,\dots,\alpha_{p+1}} = \sum_{j=0}^{p+1} (-1)^j s_{\alpha_0,\dots,\hat{\alpha}_j,\dots,\alpha_{p+1}}\Big|_{U_{\alpha_0,\dots,\alpha_{p+1}}}$$

where $\hat{\alpha}_j$ means the index α_j is omitted. Then $\delta\delta = 0$ and so we can define the cohomology groups

$$H^p(\mathcal{U},\mathcal{S}) = \{s \in C^p(\mathcal{U},\mathcal{S}) : \delta s = 0\}/\delta[C^{p-1}(\mathcal{U},\mathcal{S})].$$

Lemma 17. *Assume that the sheaf $\mathcal{S}$ is closed under multiplication by smooth functions. That is if $s \in \mathcal{S}(U)$ and f is smooth function on U, then $fs \in \mathcal{S}(U)$ for any open $U \subseteq M$. Then for any locally finite open cover $\mathcal{U}$ of M and any $p \geqslant 1$*

$$H^p(\mathcal{U},\mathcal{S}) = 0.$$

Problem 16. Prove this. *Hint:* Let $\{\rho_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to $\mathcal{U} = \{U_\alpha\}_{\alpha\in A}$. Let $s \in C^p(\mathcal{U},\mathcal{S})$ with $\delta s = 0$. Define $\tau \in C^{p-1}(\mathcal{U},\mathcal{S})$ by

$$\tau_{\alpha_0,\dots,\alpha_{p-1}} = \sum_{\beta\in A} \rho_\beta s_{\beta,\alpha_0,\dots,\alpha_{p-1}}$$

where $\rho_\beta s_{\beta,\alpha_0,\dots,\alpha_{p-1}}$ extends to $U_{\alpha_0,\dots,\alpha_{p-1}}$ by zero. Now show

$$\delta\tau = s.$$

It was easier for me to see what was going on by looking at the case of $p = 2$. Let $s \in C^2(\mathcal{U},\mathcal{S})$ with

$$\delta s_{U_0,U_1,U_2,U_3} = s_{U_1,U_2,U_3} - s_{U_0,U_2,U_3} + s_{U_0,U_1,U_3} - s_{U_0,U_1,U_2} = 0$$

Note this implies

$$s_{U_0,U_1,U_2} = s_{U_3,U_1,U_2} - s_{U_3,U_0,U_1} + s_{U_3,U_0,U_1}$$

Then τ is

$$\tau_{U_0,U_1} = \sum_V \rho_V s_{V,U_0,U_1}.$$

and you should verify the following calculation works

$$\begin{aligned}(\delta\tau)_{U_0,U_1,U_2} &= \tau_{U_1,U_2} - \tau_{U_0,U_1} + \tau_{U_0,U_1}\\ &= \sum_V \rho_V s_{V,U_1,U_2} - \sum_V \rho_V s_{V,U_0,U_1} + \sum_V \rho_V s_{V,U_0,U_1}\\ &= \sum_V \rho_V (s_{V,U_1,U_2} - s_{V,U_0,U_1} + s_{V,U_0,U_1})\\ &= \sum_V \rho_V s_{U_0,U_1,U_2}\\ &= s_{U_0,U_1,U_2}.\end{aligned}$$

This shows that for $p \geqslant 0$ that $s \in C^p(\mathcal{U}, \mathcal{S})$ and $\delta s = 0$ implies $s = \delta\tau$ for some τ. That is $H^p(\mathcal{U}, \mathcal{S}) = 0$. □

As the sheaf cohomology groups $H^p(M, \mathcal{S})$ are defined in terms of direct limits of the cohomology coming from open covers that last lemma implies

Proposition 18. *Assume that the sheaf $\mathcal{S}$ is closed under multiplication by smooth functions. Then for any $p \geqslant 1$*

$$H^p(M, \mathcal{S}) = 0.$$

□

Let $\mathcal{A}^k(M)$ be the sheaf of smooth sections of $A^p(M)$. Then $\mathcal{A}^k(M)$ is closed under multiplication by smooth functions. Thus the last proposition gives us

Theorem 19. *For all $k \geqslant 0$ and $p \geqslant 1$*

$$H^p(M, \mathcal{A}^k) = 0.$$

□

p Let let $\mathcal{Z}$ be the sheaf defined on M by

$$\mathcal{Z}^p(U) = \{\alpha \in \mathcal{A}^p(U) : d\alpha = 0\}.$$

That is $\mathcal{Z}^p$ is the sheaf of closed p forms. This sheaf is not closed under multiplication by smooth function for if $d\alpha = 0$ and f is a smooth function, then

$$d(f\alpha) = df \wedge \alpha + f d\alpha = df \wedge \alpha$$

and this need not be zero. On the other hand for $p \geqslant 1$ the Poincaré lemma can be rephrased as saying the following sequence is exact:

$$0 \to \mathcal{Z}^{p-1} \to \mathcal{A}^p \to \mathcal{Z}^p \to 0.$$

The long exact sequence for this sequence gives that for $p \geqslant 1$

$$H^{p-1}(M, \mathcal{A}^k) = 0 \to H^{p-1}(M, \mathcal{Z}^k) \to H^p(M, \mathcal{Z}^{k-1}) \to 0 = H^p(M, \mathcal{A}^{k-1}).$$

That is for all $k \geqslant 0$ and $p \geqslant 1$ we have a natural isomorphism

$$H^{p-1}(M, \mathcal{Z}^k) \cong H^p(M, \mathcal{Z}^{k-1})$$

We also have a short exact sequence

$$0 \to \mathbb{R} \to \mathcal{A}^0 \to \mathcal{Z}^1 \to 0.$$

and from the long exact sequence for this we get for $p \geqslant 1$ that

$$H^{p-1}(M, \mathcal{A}^0) = 0 \to H^{p-1}(M, \mathcal{Z}^1) \to H^p(M, \mathbb{R}) \to 0 = H^p(M, \mathcal{A}^0)$$

and thus a natural isomorphism

$$H^p(M, \mathbb{R}) \cong H^{p-1}(M, \mathcal{Z}^1).$$

We can now string all these isomorphisms together

$$\begin{aligned} H^p(M, \mathbb{R}) &\cong H^{p-1}(M, \mathcal{Z}^1) \\ &\cong H^{p-2}(M, \mathcal{Z}^2) \\ &\cong H^{p-3}(M, \mathcal{Z}^3) \\ &\ \ \vdots \\ &\cong H^1(M, \mathcal{Z}^{p-1}) \\ &\cong H^0(M, \mathcal{Z}^p)/\delta H^0(M, \mathcal{A}^{p-1}) \\ &= Z^p(M)/dA^{p-1}(M) \\ &= H^p_{\mathrm{dR}}(M). \end{aligned}$$

where δ is the connecting homomorphism from the long exact sequence. Therefore we have proven

Theorem 20 (de Rham's Theorem)**.** *For any smooth manifold M there is a natural isomorphism of graded algebras*

$$H^*_{\mathrm{dR}}(M) \cong H^*(M, \mathbb{R})$$

where $H^*(M, \mathbb{R})$ is the Čech cohomology of the constant sheaf (which is isomorphic to the singular cohomology of M with real coefficients). $\square$

So far this all has had little to do with complex manifolds. For each $p \geqslant 0$ let $\bigwedge^p_{\mathbb{C}}(M)$ be the real vector bundle with fiber at $x \in M$

$$\bigwedge\nolimits^P_{\mathbb{C}}(M)_x := \mathbb{C} \oplus_{\mathbb{R}} \bigwedge\nolimits^p(M)_x.$$

Put in somewhat more concrete terms, each fiber is the set of p-linear (over $\mathbb{R}$) alternating functions from the tangent space to $\mathbb{C}$. Let $A^p_{\mathbb{C}}(M)$ be the set of all smooth sections of $\bigwedge^p_{\mathbb{C}}(M)$. In lowbrow terms elements of $\bigwedge^p_{\mathbb{C}}(M)$ are of the form

$$\alpha = \alpha_0 + i\alpha_1$$

where α_0 and α_1 are ordinary (i.e. real valued) differential forms. Let J be the almost complex structure on M. That is for each $x \in M$ we have that $T\colon T(M)_x \to T(M)_x$ is multiplication by $i = \sqrt{-1}$. Thus $J^2 = -I$ (with I the identity map) and a real leaner map $\alpha\colon T(M)_x \to \mathbb{C}$ is complex linear if and only if

$$\alpha(JX) = i\alpha(X)$$

for $X \in T(M)_x$.

Problem 17. Let $f\colon M \to \mathbb{C}$ be a smooth function. Show that df being complex linear (that is $df(JX) = idf(X)$ for all vectors X) is equivalent to the Cauchy-Riemann equations. □

Given a real linear map $\alpha\colon T(M)_x \to \mathbb{C}$ we can write it as

$$\begin{aligned}\alpha(X) &= \frac{1}{2}\left(\alpha(X) - i\alpha(JX)\right) + \frac{1}{2}\left(\alpha(X) + i\alpha(JX)\right)\\ &= \alpha_{1,0}(X) + \alpha_{0,1}(X)\end{aligned}$$

where $\alpha_{1,0}$ as defined here is complex linear and $\alpha_{0,1}$ is conjugate linear (that $\alpha_{0,1}(JX) = -i\alpha_{0,1}(X)$). This calculation shows that every complex valued one form, that is a element of $A^1_{\mathbb{C}}(M)$ uniquely decomposes as the some of a complex linear (that is $(1,0)$ form) and a conjugate linear linear (a $(0,1)$-form). That is we have a direct sum

$$T^*_{\mathbb{C}} = \bigwedge\nolimits^1_{\mathbb{C}}(M) = A^{1,0}(M) \oplus A^{0,1}(M)$$

where $A^{1,0}(M)$ are the complex linear one forms and $A^{0,1}(M)$ are the conjugate linear one forms. In terms of a complex linear coordinate system $z^1, z^1, \ldots, z^n$ on M (where $z^j = x^j + iy^j$) the elements of $A^{1,0}(M)$ locally look like

$$\alpha = a_1\, dz^1 + a_2\, dz^2 + \cdots + a_n\, dz^n.$$

where $a_1, \ldots, a_n$ are smooth complex valued functions and

$$dz^j = dx^j + i\, dy^j.$$

Likewise the elements of $A^{0,1}(M)$ locally look like

$$\alpha = a_1\, d\overline{z}^1 + a_2\, d\overline{z}^2 + \cdots + a_n\, d\overline{z}^n.$$

With $dz^j = dx^j - idy^j$ and $a_1, \ldots, a_n$ are smooth complex valued functions.

Problem 18. If you have not done so before, you should check that the real linear map $dz^j\colon T(M)_x \to \mathbb{C}$ is complex linear and that $d\overline{z}^j\colon T(M)_x \to \mathbb{C}$ is conjugate linear. □

Let $A^{p,q}(M)$ be the sections of the bundle

$$A^{p,q}(M) = A^{1,0}(M) \otimes A^{0,1}(M).$$

Sections of $A^{2,0}(M)$ are of the form

$$\sum_{j<k} a_{jk}\, dx^j \wedge dz^k.$$

Sections of $A^{1,1}(M)$ are of the form

$$\sum_{j,k} a_{ij}\, dz^j \wedge d\overline{z}^k$$

and finally it will be no surprise that $(0,2)$ forms are of the form

$$\sum_{j<k} a_{jk}\, d\overline{z}^j \wedge d\overline{z}^k.$$

where in all of these formulas the a_{jK}'s are smooth complex valued functions. In general if $\mathcal{I}(p)$ is the set of tuples $(i_1, i_2, \ldots, i_p)$ with $1 \leqslant i_1 < i_2 < \cdots < i_p \leqslant n$ then elements of $A^{p,q}(M)$, which are called (p,q)-***forms*** on M, are of the form

$$\sum_{J\in\mathcal{I}(p), K\in\mathcal{I}(q)} a_{IJ}\, dz^J \wedge d\overline{z}^K$$

where $dz^J = dz^{j_1} \wedge dz^{j_2} \wedge \cdots \wedge dz^{j_p}$ and $d\overline{z}^K = d\overline{z}^{k_1} \wedge d\overline{z}^{k_2} \wedge \cdots \wedge d\overline{z}^{k_q}$.

Let $z^1, \ldots, z^n$ be complex coordinates on with $z^j = x^j + iy^j$. Then the exterior derivative of a smooth function $f\colon M \to \mathbb{C}$ is

$$df = \sum_{j=1}^{n} \left(\frac{\partial f}{\partial x^j}\, dx^j + \frac{\partial f}{\partial y^j}\, dy^j \right).$$

We define the partial derivatives with respect to z^j and $\overline{z}^j$ as usual:

$$\frac{\partial}{\partial z^j} := \frac{1}{2}\left(\frac{\partial}{\partial x^j} - i\frac{\partial}{\partial y^j} \right), \qquad \frac{\partial}{\partial \overline{z}^j} = \left(\frac{\partial}{\partial x^j} + i\frac{\partial}{\partial y^j} \right).$$

Proposition 21. *For a smooth complex valued function, f, on a complex manifold, M, the exterior derivative is given by*

$$df = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial z^k}\, dz^k + \frac{\partial f}{\partial \overline{z}^k}\, d\overline{z}^k \right).$$

Problem 19. If you have near done this calculation before, do it now. □

This suggests defining differential operators ∂ and $\overline{\partial}$ defined in local coordinates by

$$\partial f = \sum_{k=1}^{n} \frac{\partial f}{\partial z^k}\, dz^k, \qquad \overline{\partial} f = \sum_{k=1}^{n} \frac{\partial f}{\partial \overline{z}^k}\, d\overline{z}^k,$$

It can easily be verified that this definition is independent of the complex coordinate used to define it. This also follows from the following.

Proposition 22. *Let f be a smooth complex valued function on the complex manifold M. Then ∂f and $\overline{\partial} f$ are the (real) linear maps given by*

$$\partial f(X) = \frac{1}{2}\left(df(X) - i\, df(JX) \right), \qquad \overline{\partial} f(X) = \frac{1}{2}\left(df(X) + i\, df(X) \right).$$

Thus

$$df = \partial f + \overline{\partial} f$$

and this is the decomposition of df into the sum of complex linear and a conjugate linear maps.

Problem 20. This is anther case where if you have not done this before, you should do it now. □

We now extend the definitions of ∂ and $\overline{\partial}$ to $A^*(M)$ in the natural way. That is if

$$\alpha = \sum_{J,K} a_{JK}\, dz^J \wedge d\overline{z}^K$$

then

$$\partial\alpha = \sum_{J,K} \partial a_{JK} \wedge dz^J \wedge d\overline{z}^K, \qquad \overline{\partial}\alpha = \sum_{J,K} \overline{\partial} a_{JK} \wedge dz^J \wedge d\overline{z}^K.$$

With this definition we still have that

$$d = \partial + \overline{\partial}$$

holds on $A^*(M)$. From this definition it is clear that

$$\partial A^{p,q}(M) \subseteq A^{p+1,q}(M), \qquad \text{and} \qquad \overline{\partial} A^{p,q}(M). \subseteq A^{p,q+1}(M)$$

Proposition 23. *The following hold*

$$\partial\partial = 0, \qquad \partial\overline{\partial} + \overline{\partial}\partial = 0, \qquad \overline{\partial}\,\overline{\partial} = 0.$$

Problem 21. Prove this. *Hint:* From $d = \partial + \overline{\partial}$ and $dd = 0$ we have

$$0 = dd = (\partial + \overline{\partial})(\partial + \overline{\partial}) = \partial\partial + (\partial\overline{\partial} + \overline{\partial}\partial) + \overline{\partial}\,\overline{\partial}.$$

Let α be a (p,q) form. Then

$$0 = \partial\partial\alpha + (\partial\overline{\partial} + \overline{\partial}\partial)\alpha + \overline{\partial}\,\overline{\partial}\alpha$$

and

$$\partial\partial\alpha \in A^{p+2,q}(M), \quad (\partial\overline{\partial} + \overline{\partial}\partial)\alpha \in A^{p+1,q+1}(M), \quad \overline{\partial}\,\overline{\partial}\alpha \in A^{p,q+2}(M).$$

Now use that the spaces $A^{p+2,q}(M)$, $A^{p+1,q+1}(M)$ and $A^{p,q+2}(M)$ are linearly independent. $\square$

Note if $f\colon M \to \mathbb{C}$, then $\overline{\partial} f = 0$ if and only if f is holomorphic. More generally a ***holomorphic*** p***-form*** is a form $\alpha \in A^{p,0}(M)$ that satisfies $\overline{\partial} f = 0$.

We now list the sheaves that will be of use to us. First if M is any smooth manifold then we have

$\mathbb{Z}(U)$ = locally constant $\mathbb{Z}$ valued functions on U.

$\mathbb{R}(U)$ = locally constant $\mathbb{R}$ valued functions on U.

$\mathbb{C}(U)$ = locally constant $\mathbb{C}$ valued functions on U.

$\mathbb{C}^*(U)$ = locally constant $\mathbb{C}^*$ valued functions on U.

$\mathcal{C}^\infty(U)$ = smooth $\mathbb{R}$ (or $\mathbb{C}$ depending on context) valued functions

$\mathcal{A}^p(U)$ = smooth p forms on U.

$\mathcal{Z}^p(U)$ = smooth closed forms U.

If M is a complex manifold we get some more sheave defined by the complex structure.

$$\begin{aligned}
\mathcal{O}(U) &= \text{holomorphic functions on } U.\\
\mathcal{O}^*(U) &= \text{group of nonzero holomorphic functions on } U.\\
\Omega^p(U) &= \text{holomorphic } p \text{ forms on } U.\\
\mathcal{A}^{p,q}(U) &= \text{smooth } (p,q) \text{ forms on } U.\\
\mathcal{Z}^{p,q}_{\overline{\partial}}(U) &= (p,q)\text{-forms } \alpha \text{ with } \overline{\partial}\alpha = 0
\end{aligned}$$

Definition 24. A sheaf, $\mathcal{S}$, on a smooth manifold, M, is ***fine*** if is closed under multiplication by smooth functions in the sense of Lemma 17. (This is not the standard definition, which is that the sheaf admits partitions of unity. On smooth manifolds and with the sheaves we will be considering the definition here is easier to verify.) □

Proposition 25. *If M is a complex manifold the following sheaves*

$$\mathcal{C}^\infty,\ \mathcal{A}^p,\ \mathcal{A}^{p,q}$$

are fine and therefore for $k \geqslant 1$

$$H^k(M,\mathcal{C}^\infty) = 0,\ H^k(M,\mathcal{A}^p) = 0,\ H^k(M,\mathcal{A}^{p,q}) = 0.$$

Problem 22. Prove this. *Hint:* That these sheaves are closed under multiplication by smooth functions is more or less clear. To conclude that the cohomology groups vanish use Proposition 18. □

Having sheaves, $\mathcal{S}$, that have $H^p(M,\mathcal{S}) = 0$ makes us want to put then into short exact sequences to get bunches of isomorphisms. We have already seen one example of this using that the sequences

$$0 \longrightarrow \mathbb{R} \xrightarrow{\text{incl.}} \mathcal{A}^0 \xrightarrow{d} \mathcal{Z}^1 \longrightarrow 0$$

and for $p \geqslant 2$

$$0 \longrightarrow \mathcal{Z}^{p-1} \xrightarrow{\text{incl.}} \mathcal{A}^{p-1} \xrightarrow{d} \mathcal{Z}^p \longrightarrow 0$$

are exact which was used in the proof of the de Rham Theorem. That these are exact is just a restatement of the Poincaré lemma.

There is also a Poincaré lemma for the $\overline{\partial}$ operator. Recall that a ***polydisk*** in $\mathbb{C}^n$ is a set of the form

$$U = D(a_1,r_1) \times D(a_2,r_2) \times \cdots \times D(a_n,r_n)$$

where $D(a_k,r_k) = \{z \in \mathbb{C} : |z - a_k| < r_k\}$.

Proposition 26. *If U is a polydisk, $p \geqslant 1$, and α is a (p,q) form on U with $\overline{\partial}\alpha = 0$, then there is a $(p-1,q)$ for β with $\overline{\partial}\beta = \alpha$.*

Proof. We will come back to this. □

We can translate this into a statement about sheaves which we put in a somewhat long winded form.

Proposition 27. *Let M be a complex manifold. Then the sequence*

$$0 \longrightarrow \mathcal{O} \xrightarrow{\textit{incl.}} \mathcal{A}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{Z}^{1,0} \longrightarrow 0$$

(where $\mathcal{A}^{0,0}$ can be viewed as $\mathcal{A}^{0,0} = \mathcal{C}^{\infty}$ the sheaf of smooth complex valued functions) is exact. For $p \geqslant 0$

$$0 \longrightarrow \Omega^p \xrightarrow{\text{incl.}} \mathcal{A}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,1}_{\overline{\partial}} \longrightarrow 0$$

is exact. For $p \geqslant 1$ the sequence

$$0 \longrightarrow \mathcal{Z}^{p-1,0}_{\overline{\partial}} \xrightarrow{\text{incl.}} \mathcal{A}^{p-1,0} \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,0}_{\overline{\partial}} \longrightarrow 0$$

are exact. For all p, q

$$0 \longrightarrow \mathcal{Z}^{p,q}_{\overline{\partial}} \xrightarrow{\text{incl.}} \mathcal{A}^{p,q} \xrightarrow{\overline{\partial}} \mathcal{Z}^{p,q+1}_{\overline{\partial}} \longrightarrow 0$$

is exact. □

We also have the $\overline{\partial}$ version of the de Rham groups:

$$H^{p,q} = Z^{p,q}_{\overline{\partial}}(M) \Big/ \overline{\partial}\left[A^{p,q-1}(M)\right].$$

Theorem 28. *For a complex manifold there is a natural isomorphism*

$$H^q(M, \Omega^p) \cong H^{p,q}_{\overline{\partial}}(M).$$

Problem 23. Give a proof of this along the lines of the proof of Theorem 20.

□