Mathematics 739 Homework 4: Hodge Theory.

We first review some functional analysis. Let X and Y inner product spaces (we are not assuming that they are complete). Let $P\colon X \to Y$ be a linear map. Then a linear map $P^*\colon Y \to X$ is an ***adjoint*** to P if

$$\langle Px, y\rangle_Y = \langle x, P^* y\rangle_X$$

In what we have in mind here most of the operators will be differential operators and the adjoints are usual found by integration by parts. Here is an example. Let $X = Y$ be the space of elements, f, of $C^\infty([0,1])$ with $f(0) = f(1) = 0$ with the usual L^2 norm. Let $Pf = f'$ be the derivative of f. Then for any $g \in Y$ we have

$$\begin{aligned}\langle Pf, g\rangle &= \int_0^1 f'(t)g(t)\,dt \\ &= f(t)g(t)\Big|_0^1 - \int_0^1 fg'(t)\,dt \\ &= -\int_0^1 fg'(t)\,dt\end{aligned}$$

and in this case we have $P^*g = -g'$.

Here is the setup for a what might be thought of as "pre Hodge Theory". Let

$$A \xrightarrow{d_1} B \xrightarrow{d_2} C$$

be a sequence of inner product spaces and linear maps such that

$$d_2 \circ d_1 = 0.$$

Then we have the cohomology group

$$H = \ker(d_2)/d_1[A_1].$$

In general a cohomology class $[\alpha] \in H$ will have infinitely many elements. Our goal to so choose a single element of the class $[\alpha]$ in a canonical way. Since we are working with inner product spaces, choosing the element of a class that has shortest length is a natural idea.

So let $[\beta] \in H^2$ and assume that it has an element β_0 that has minimal length of all elements of the class $[\beta_0] = [\beta]$. Then for any $\alpha \in A$ and $t \in \mathbb{R}$ the element $\beta_0 + td_1\alpha$ is in the same cohomology class as β_0 and therefore

$$\|\beta_0 + t\alpha\|^2 = \langle \beta_0 + td_1\alpha, \beta_0 + td_1\alpha\rangle = \|\beta_0\|^2 + 2t\langle d_1\alpha, \beta_0\rangle + t^2\|d_1\alpha\|^2$$

has a minimum at $t = 0$. Therefore by the first derivative test

$$\langle d_1\alpha, \beta_0\rangle = \langle \alpha, d_1^*\beta_0\rangle = 0.$$

As this holds for all $\alpha \in A$ this implies

$$d_1^*\beta_0 = 0.$$

Lemma 1. *Let* $\beta_0 \in B$ *satisfy*

$$d_2\beta_0 = 0 \quad \textit{and} \quad d_1^*\beta_0 = 0. \tag{1}$$

Then β_0 *is the unique element of its cohomology class* $[\beta_0]$ *of minimum norm.*

Problem 1. Prove this. *Hint:* Every element of $[\beta_0]$ is of the form $\beta_0 + d_1\alpha$ for some $\alpha \in A$. Use that $d_1^*\beta_0 = 0$ to show

$$\|\beta_0 + d_1\alpha\|^2 = \|\beta_0\|^2 + \|d_1^*\alpha\|^2 \geqslant \|\beta_0\|^2.$$

To show uniqueness assume that β_1 is in the same cohomology class and β_0 and also has minimum norm. Show that

$$\beta = \frac{1}{2}(\beta_0 + \beta_1)$$

is in the same cohomology class as β_0 and β_1 and if $\beta_0 \neq \beta_1$ it has smaller norm. □

This shows that find finding an element of a cohomology $[\beta] \in H$ is equivalent to finding a β_0 in the class that satisfies the two equations of (1). It is possible to reduce these two equations to a single equation by introducing the ***Hodge Laplacian***

$$\Delta := d_1 d_1^* + d_2^* d_2.$$

Lemma 2. *With the notation of Lemma 1 for* $\beta_0 \in B$ *the two equations*

$$d_2\beta_0 = 0 \quad \textit{and} \quad d_1^*\beta_0 = 0.$$

are equivalent to the equation

$$\Delta\beta_0 = 0.$$

Proof. Prove this. *Hint:* One way to start is by showing that

$$\langle \Delta\beta_0, \beta_0 \rangle = \|d_1^*\beta_0\|^2 + \|d_2\beta_0\|^2.$$

□

To relate this to the de Rham cohomology $H^*_{\mathrm{dR}}(M)$ we need to make the spaces A^k into inner product spaces. The first step is to put an inner product on each tangent space to M. A ***Riemannian metric*** on a smooth manifold, M, is choice for each $x \in M$ of an inner product $g_x(\,,\,)$ on TM_x. That is $g_x(\,,\,)\colon TM_x \times TM_x \to \mathbb{R}$ is a symmetric positive definite bilinear. Note if $x^1, x^2, \ldots, x^n$ is a coordinate system on an open set U in M, then U has a Riemannian metric:

$$g = (dx^1)^2 + (dx^2)^2 + \cdots + (dx^n)^2.$$

(This is the inner product that makes the coordinate vectors $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ into an orthonormal basis.) This shows that every point of M has an open neighborhood that has a Riemannian metric.

Proposition 3. *Every smooth manifold has a Riemannian metric.*

Problem 2. Prove this. *Hint:* This is a special case of the meta theorem that any pointwise construction on vector spaces that is closed under convex combinations is can be defined on a manifold by use of a partition of unity. In this case we can find the Riemannian metric on M of the form $\sum_{\alpha\in A}\rho_\alpha g_\alpha$ where $\{\rho_\alpha\}_{\alpha\in A}$ is a partition of unity and each g_α is a Riemannian metric defined on some open subset U_α of M. $\square$

Let V be a n-dimensional real inner product space with a fixed orientation. finite dimensional real vector space with an inner product $\langle\,,\rangle$. Then each of the exterior powers $\bigwedge^k(V)$ also has a natural inner defined on decomposable by

$$\langle v_1\wedge v_2\wedge\cdots\wedge v_k, w_1\wedge w_2\wedge\cdots\wedge w_k\rangle = \det\big([\langle v,w_j\rangle]_{i,j=1}^k\big).$$

As $\dim V = n$ the space $\bigwedge^n(V)$ is one dimensional. Let $e_1,e_2,\ldots,e_n$ be an oriented orthonormal basis of V and set $\omega = e_1\wedge e_2\wedge\cdots\wedge e_n$. This is independent of the choice of the oriented orthonormal bases used to define it. Let $1\leqslant k\leqslant n-1$ and let $\beta\in\bigwedge^{n-k}(V)$. The if $\alpha\in\bigwedge^k(V)$ the product $\alpha\wedge\beta$ is in $\bigwedge^n(V)$ which is one dimensional. Therefore for some scalar $b(\alpha,\beta)\in\mathbb{R}$ we have

$$\alpha\wedge\beta = b(\alpha,\beta)\omega.$$

It is clear that $b(\alpha,\beta)$ is linear function of each to its arguments. Thus for fixed β the map $\alpha\mapsto b(\alpha,\beta)$ is linear and $\bigwedge^k(V)$ is an inner product. But every linear functional on a finite dimensional vector space can be represented as an inner product with a vector. Thus there is a unique vector $\star\beta\in\bigwedge^k(V)$ such that

$$\alpha\wedge\beta = \langle\alpha,\star\beta\rangle\omega.$$

The map $\beta\mapsto\star\beta$ is the ***Hodge star*** and is a linear map from $\bigwedge^{n-k}(V)$ to $\bigwedge^k(V)$. The equation above can be taken as the definition of $\star\beta$. We use the convention that $\bigwedge^0(V)=\mathbb{R}$ and $\star\colon\bigwedge^0(V)\to\bigwedge^n(V)$ and $\star\colon\bigwedge^n(V)\to\bigwedge^0(V)$ are given by

$$\star 1 = \omega, \qquad \star\omega = 1.$$

We now wish to compute $\star\star\alpha$. While one would think this is just a trivial chase through the definition, looking a some books and googleing and reading the question and answers on stack overflow has lead met to conclude that it is tricker than that. Here is a proof that use symmetry in a nice way.[1] Recall that if $A\colon V\to V$ is linear, then it defines a map $\bigwedge^k(A)\colon\bigwedge^k(V)\to\bigwedge^k(V)$ by defining it on decomposable element as

$$\textstyle\bigwedge^k(A)(v_1\wedge v_2\wedge\cdots\wedge v_k) = (Av_1)\wedge(Av_2)\wedge\cdots\wedge(Av_k)$$

Lemma 4. *Let $A\in\mathrm{SO}(V)$ (that is $A\colon V\to V$ is a linear map that preserves both the inner product and the orientation). Then A commutes with $\star$ in the sense that*

$$\star\textstyle\bigwedge^{n-w}(A) = \bigwedge^k(A)\,\star.$$

[1] I took the basic idea used here is from the notes *The Hodge Stat Operator* by Rich Schartz on his web page `https://www.math.brown.edu/~res/M114/`

Problem 3. Prove this by verifying the following calculation. (This does involve knowing a bit of multi-linear algebra.)

$$\begin{aligned}
\langle \alpha, \star \textstyle\bigwedge^{n-k}(A)\beta \rangle \omega &= \alpha \wedge (\textstyle\bigwedge^{n-k}(A)\beta) \\
&= \textstyle\bigwedge^{n}(A^{-1}) \Big(\alpha \wedge (\textstyle\bigwedge^{n-k}(A)\beta) \Big) \quad \text{(as } \det(A^{-1}) = 1) \\
&= \Big(\textstyle\bigwedge^{k}(A^{-1})\alpha \Big) \wedge \beta \\
&= \langle \textstyle\bigwedge^{k}(A^{-1})\alpha, \star\beta \rangle \omega \\
&= \langle \alpha, \textstyle\bigwedge^{k}(A) \star \beta \rangle \omega \qquad \text{(in } SO \text{ transpose = inverse)}
\end{aligned}$$

This holds for all $\alpha \in \bigwedge^k(V)$ and therefore $\star \bigwedge^{n-k}(A)\beta = \bigwedge^k(A) \star \beta$ □

Lemma 5. *Let $P\colon \bigwedge^k(V) \to \bigwedge^k(V)$ be a linear map that commutes with $\bigwedge^k(A)$ for all $A \in \mathrm{SO}(V)$. Then $P = \lambda I$ for some real number λ.*

Problem 4. Prove this. *Hint:* This is a standard fact from representation theory. □

Proposition 6. *On $\bigwedge^{n-k}(V)$ the Hodge star satisfies $\star\star\beta = (-1)^{k(n-k)}\beta$.*

Problem 5. Prove this. *Hint:* Define $P\colon \bigwedge^{n-k}(V) \to \bigwedge^{n-k}(V)$ by $P\beta := \star\star\beta$. By Lemma 4 we have that $P\bigwedge^{n-k}(A) = P\bigwedge^{n-k}(A)$ for all $A \in \mathrm{SO}(V)$. Then by Lemma 5 we have $P = \lambda I$ for for λ. To compute λ show that if $e_1, \ldots, e_n$ is a an oriented orthonormal normal basis of V, then

$$\star\star e_{k+1} \bigwedge e_{k+2} \wedge \cdots \wedge e_n = (-1)^{k(n-k)} e_{k+1} \wedge e_{k+2} \wedge \cdots \wedge e_n$$

and thus $\lambda = (-1)^{k(n-k)}$. □

Let (M, g) be a oriented Riemannian. That is M is a smooth manifold with an orientation and g is a smooth Riemannian metric on M. Then each cotangent space $T^*(M)_x$ is an inner product space (the inner product on $T(M)_x$ determines an inner product on $T^*(M)_x$) and so we can more our definition of the Hodge star to forms on (M, g). Therefore if $\alpha \in A^k(M)$ is a smooth k form on M, then $\star\alpha$ is a smooth $(n-k)$-form. And $\star 1 = \omega$ is the volume form on M.

Problem 6. As practice in working with these definitions show that on $\mathbb{R}^2$ with the standard flat metric $g = (dx)^2 + (dy)^2$ and the usual orientation that

$$\begin{aligned}
\star 1 &= dx \wedge dy \\
\star dx &= dy \\
\star dy &= -dx \\
\star(dx \wedge dy) &= 1.
\end{aligned}$$

□

Problem 7. For the flat metric $(dx)^2 + (dy)^2 + (dz)^2$ on $\mathbb{R}^3$ with its usual orientation show

$$\begin{aligned} \star 1 &= dx \wedge dy \wedge dz \\ \star dx &= dy \wedge dz \\ \star dy &= -dx \wedge dz \\ \star dz &= dx \wedge dy \\ \star(dx \wedge dy) &= dz \\ \star(dx \wedge dz) &= -dy \\ \star(dy \wedge dz) &= dx \\ \star(dx \wedge dy \wedge dz) &= 1. \end{aligned}$$

□

Proposition 7. *The adjoint of the exterior derivative* $d\colon A^k(M) \to A^{k+1}(M)$ *is*

$$d^* = (-1)^{(k+1)(n-k)} \star d \star .$$

Problem 8. Prove this by checking to see if I have the signs correct in the following calculation, which uses Stokes' Theorem and integration by parts. Let $\alpha \in A^k(M)$ and $\beta \in A^{k+1}(M)$.

$$\begin{aligned} \langle \alpha, d^*\beta \rangle_{L^2} &= \langle d\alpha, \beta \rangle_{L^2} \\ &= \int_M \langle d\alpha, \beta \rangle \omega \\ &= (-1)^{(k+1)(n-k-1)} \int_M \langle d\alpha, \star\star\beta \rangle \omega \\ &= (-1)^{(k+1)(n-k-1)} \int_M d\alpha \wedge \star\beta \\ &= (-1)^{(k+1)(n-k-1)} \int_M \Big(d(\alpha \wedge \star\beta) - (-1)^k \alpha \wedge d(\star\beta) \Big) \\ &= (-1)^{(k+1)(n-k-1)} (-1)^{k+1} \int_M \alpha \wedge d(\star\beta) \\ &= (-1)^{(k+1)(n-k)} \int_M \langle \alpha, \star d(\star\beta) \rangle \omega \\ &= \langle \alpha, (-1)^{(k+1)(n-k)} \star d(\star\beta) \rangle_{L^2} \end{aligned}$$

as required. □

In the last proposition we have computed d^* with domain $A^{k+1}(M)$. In comparing with other sources it is nice to have it with domain $A^k(M)$, which is just done by replacing k by $k-1$.

Corollary 8. *The adjoint of* $d\colon A^{k-1}(M) \to A^k(M)$ *is*

$$d^* = (-1)^{n(n-k)+1} \star d \star$$

□

The following is the main result of Hodge Theory for smooth compact Riemannian manifolds.

Theorem 9 (The Hodge Theorem)**.** *Let (M, g) be a smooth compact oriented Riemannian manifold. Then each each de Rham cohomology $[\alpha] \in H^*(M)$ contains a unique harmonic form α_0. That is there is a unique form $\alpha_0 \in [\alpha]$ with*

$$\Delta\alpha_0 = (d^*d + dd^*)\alpha_0 = 0.$$

Since M (so that we can integrate by parts and get $\langle \Delta\alpha_0, \alpha_0\rangle_{L^2} = \|d\alpha_0\|_{L^2}^2 + \|d^*\alpha_0\|_{L^2}^2$) being harmonic is equivalent to the two equations

$$d\alpha_0 = 0 \qquad \text{and} \qquad d \star \alpha_0 = 0.$$

□

One application of this is Poincaré duality for de Rham cohomology.

Theorem 10. *Let M be a smooth oriented manifold of dimension n. Then for $0 \leqslant k \leqslant n$ the pairing $b(\,,\,)\colon H^k_{\mathrm{dR}}(M) \times H^{n-k}_{\mathrm{dR}}(M) \to \mathbb{R}$ given by*

$$b([\alpha],[\beta]) = \int_M \alpha \wedge \beta$$

is nondegenerate.

Problem 9. Prove this. *Hint:* First show that definition of $b([\alpha],[\beta])$ is independent of the choice of the forms representing the classes. Saying that the bilinear form $b(\,,\,)$ is nondegenerate is saying that if for all $[\beta] \in H^{n-k}_{\mathrm{dR}}(M)$ we have $b([\alpha],[\beta]) = 0$, then $[\alpha] = 0$. (And likewise with the roles of $[\alpha]$ and $[\beta]$ interchanged.)

To prove nondegeneracy choose any smooth Riemannian metric g on M and let $\star$ be the Hodge star of this metric. Let $[\alpha] \in H^k_{\mathrm{dR}}(M)$ be a nonzero cohomology class. Let α_0 be the harmonic form in this class. Then $\star\alpha_0$ is also a harmonic form and $[\star\alpha_0] \in H^{n-k}_{\mathrm{dR}}(M)$. Then

$$b([\alpha_0],[\star\alpha_0]) = \int_M \alpha_0 \wedge \star\alpha_0 = \int_M \langle \alpha_0, \alpha_0\rangle \omega = \int_M \|\alpha_0\|^2 \, \omega > 0.$$

Use this to show the nondegeneracy. □

Let us loot at an example. Let v_1 and v_2 be linearly independent vectors in $\mathbb{R}^n$. Let

$$\mathcal{L} = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2$$

be the integral lattice generated by these vectors and let

$$M = \mathbb{R}^2/\mathcal{L}.$$

Then the Riemannian metric

$$g = (dx)^2 + (dy)^2$$

is translation invariant and therefore it descends to a metric on M.

We first look at the 0-forms on M. These can be viewed as the functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ that are periodic with respect to $\mathcal{L}$. That is for all $z \in \mathbb{R}^2$ and

$j = 1, 2$ we have $f(z + v_j) = f(z)$. If f is harmonic, then $df = 0$ which implies that f is constant. Therefore the harmonic 0-forms on M are just the constants.

On M the one forms are of the form

$$\alpha = P\,dx + Q\,dy$$

where P and Q are functions that are periodic with respect to $\mathcal{L}$. Then

$$\star\alpha = P\,dy - Q\,dx.$$

For α to be harmonic we need

$$d\alpha = \left(-\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x}\right) dx \wedge dy = 0$$
$$d \star \alpha = \left(\frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x}\right) dx \wedge dy = 0.$$

Together these imply

$$\frac{\partial P}{\partial x} = \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{\partial Q}{\partial y} = 0$$

which implies that P and Q are constant.

Finally the 2 forms are of the form

$$\alpha = f\,dx \wedge dy$$

where f is periodic with respect to $\mathcal{L}$. If this is harmonic we have

$$0 = d \star f dx \wedge dy = d(f \star \omega) = (d(f\,1) = df$$

and so f is constant.

Thus on $M = \mathbb{R}^2/\mathcal{L}$ the harmonic forms are just the constant coefficient forms. Thus $H^*_{\mathrm{dR}}(M)$ is just the algebra generated by dx and dy. That is

$$H^*_{\mathrm{dR}}(M) = \mathbb{R}[dx, dy].$$

Problem 10. Let v_1, v_2, v_3 be three linearly independent vectors in $\mathbb{R}^3$ and

$$\mathcal{L} = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3$$

the integral lattice they generate. Let

$$M = \mathbb{R}^3/\mathcal{L}$$

and use the flat metric $(dx)^2 + (dy)^2 + (dz)^2$ on this manifold. Show that the harmonic forms on M are the constant coefficient forms on M and therefore

$$H^*_{\mathrm{dR}}(M) = \mathbb{R}[dx, dy, dz].$$

□

Problem 11. The last problem and example might lead you to conjecture that if α and β are harmonic forms on a smooth oriented Riemannian manifold (M, g) that $\alpha \wedge \beta$ is also harmonic and thus that the space of harmonic forms is an algebra with respect to the product $\wedge$. Here we give an example to show this is not true. Let M be a compact oriented manifold with $H^1_{\mathrm{dR}}(M) \neq 0$ and with nonzero Euler characteristic $\chi(M)$. An example would be a genus g surface, M_g, with $g \geqslant 2$ (as $\chi(M) = 2 - 2g$). Let g be any smooth Riemannian metric on M and let α be a harmonic 1-form in a nonzero homology class of $H^1_{\mathrm{dR}}(M)$. Because $\chi(M) \neq 0$ the one form α must vanish at at least one point x_0. (We will see a proof of this later in the term.) The form $\star\alpha$ is also harmonic. Show $\alpha \wedge \star\alpha$ is not harmonic. *Hint:* If $\dim(M) = n$, then $\alpha \wedge \star\alpha$ is an n-form on M. It vanishes at x_0. But the only harmonic n-forms on M are constant multiples of the volume form ω and these do not vanish at any point. □

To get a bit of practice in working with the cohomology groups, let us go back to a lattice $\mathcal{L}$ in $\mathbb{R}^2$. Assume that

$$\mathcal{L} = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2.$$

Then for $j = 1, 2$ let $\gamma_j\colon [0,1] \to M := \mathbb{R}^2/\mathcal{L}$ be

$$\gamma_j(t) = \langle tv_j \rangle = \text{coset of } tv_j \text{ in } M = \mathbb{R}^2/\mathcal{L}.$$

Since $v_j \in \mathcal{L}$ this is a closed curve in M and in fact the homology classes represented by γ_1 and γ_2 are a basis for the homology group $H_1(M, \mathbb{Z})$. Call a cohomology class $[\alpha] \in H^1_{\mathrm{dR}}(M)$ an ***integral cohomology class*** if and only if

$$\int_\gamma \alpha \in \mathbb{Z}$$

for all closed curves γ in M. Since γ_1 and γ_2 represent a basis for $H_1(M, \mathbb{Z})$ this is equivalent to

$$\int_{\gamma_1} \alpha, \quad \int_{\gamma_2} \alpha \in Z.$$

Problem 12. Let v_1 and v_2 be the vectors

$$v_1 = (v_{11}, v_{12}), \qquad v_2 = (v_{21}, v_{22}).$$

Assume these are linearly independent and let $\gamma_1, \gamma_2\colon [0,1] \to M = \mathbb{R}^2/\mathcal{L}$ be as above. Show that

$$\int_{\gamma_1} dx = v_{11}, \qquad \int_{\gamma_2} dx = v_{21}$$
$$\int_{\gamma_1} dy = v_{12}, \qquad \int_{\gamma_2} dy = v_{22}$$

and use this to show that a basis for the integral cohomology in $H^1_{\mathrm{dR}}(M)$ are the forms

$$\alpha = a_1\,dx + a_2\,dy, \qquad \beta = b_1\,dx + b_2\,dy$$

where the numbers a_1, a_2, b_1, and b_2 are given by

$$\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}^{-1}$$

□