Mathematics 739 Homework 5: Connections and Chern Classes.

Let M be a smooth manifold. We have talked about differential forms on M, where a k-form α is a smooth section of the bundle $\bigwedge^k(T^*)(M)$. Thus for each $x \in M$ we have that α_x is a k-linear alternating function on $T(M)_x$. The exterior derivative was defined on functions $f\colon M \to \mathbb{R}$ by as the linear functional $df(X) := Xf$ where we view vectors, X, as differential operators. This defines d on 0-forms. Then d is extended to k forms so that the rule

$$d(f\alpha) = df \wedge \alpha + f d\alpha$$

holds.

A slight generalization is to let V be a finite dimensional vector space over $\mathbb{R}$ and we can then define V valued k-forms, α, so that for $x \in M$ then α_x is a k-linear alternating function on $T(M)_x$ with values in V. The for a V-valued 0-form, we can still define $df(X) = Xf$, and still extend to k forms so that the basic product rule $d(f\alpha) = df \wedge \alpha + f d\alpha$ holds. Now this still works when V is a complex vector space, as V is also a real vector space.

A more far reaching generalization is to let $p\colon E \to M$ be a smooth vector bundle over M. Then the bundle

$$\textstyle\bigwedge^k(T^*(M)) \otimes E$$

has as its fiber at $x \in M$ the set of all k-linear alternating functions from $T(M)_x$ to E_x. When $k = 0$ this is isomorphic to E. We introduce what I hope is natural notation. Let U be an open subset of M

$$A^k(U, E) = \text{sections of } \textstyle\bigwedge^k(T^*(M)) \otimes E \text{ over } U.$$
$$\mathcal{A}^k(M) = \text{sheaf of sections of } \textstyle\bigwedge^k(T^*(M)) \otimes E.$$

It would now be nice if for sections, s, of E, that is elements of $A^0(M, E)$, that there was a unique way to assign an analogue of the exterior derivative ds in such a way that usual rules work. But no such luck. So we make a definition.

Definition 1. Let $p\colon E \to M$ be a smooth vector bundle over M. Then a ***connection*** (also called a ***covariant derivative***) is a $\mathbb{R}$-linear map $\nabla\colon A^0(M, E) \to A^1(M, E)$ such that for $s \in A^0(M, E)$ and f a smooth real valued function the Leibniz rule

$$\nabla(fs) = df \otimes s + f \nabla s$$

holds. □

If $X \in T(M)_x$ then we will sometimes write $\nabla s(X)$ as $\nabla_X s$ and think of this as the derivative of s in the direction of X.

At this point it is not at all clear that vector bundles need have any connections at all. Here is a start on showing that they do.

Lemma 2. *Let $E \to M$ be a vector bundle and $U \subseteq M$ an open subset where E has a frame over U. (That is there are sections $e_1, e_2, \ldots, e_m$ such that for each $x \in U$ the vectors $e_1(x), e_2(x), \ldots, e_m(x)$ form a basis for E_x. Then every section of E over U is of the form*

$$s = s^1 e_1 + s^2 e_2 + \cdots + s^m e_m.$$

for smooth uniquely determined functions $s^1, s^2, \ldots, s^m$. Define

$$\nabla s = ds^1 \otimes e_1 + ds^2 \otimes e_2 + \cdots + ds^m e_m.$$

Then ∇ is a connection on U.

Problem 1*.* Prove this. □

This shows that connections exist locally. It also shows that we can not expect connections to be unique. For different choices of local frames in the construction here will lead to different connections.

Lemma 3. *Let $E \to M$ be a smooth vector bundle and let U be an open subset of M. Let $\nabla_1, \ldots, \nabla_\ell$ be connections for E on U and let $\rho_1, \ldots, \rho_m \colon M \to \mathbb{R}$ be smooth functions with $\rho_1 + \cdots + \rho_\ell = 1$ on M. Then*

$$\nabla = \sum_{j=1}^{\ell} \rho_j \nabla_j$$

is a connection for E over U.

Problem 2*.* Prove this. □

Theorem 4. *Let $E \to M$ be a vector bundle over M. Then E has a connection.*

Problem 3*.* Use the last two lemmas and a partition of unity argument to prove this. □

If $E \to M$ is a vector bundle over M, then a ***local frame*** for E is a set of smooth sections $e_1, \ldots, e_m$ defined on an open subset U of M such that for each $x \in U$ the vectors $e_1(x), \ldots, e_m(x)$ are a basis of E_x. If $e_1', \ldots, e_m'$ is anther local frame for E defined over U then these are related by

$$e_j' = \sum_k a_{jk} e_k.$$

for uniquely determined functions smooth functions a_j^k. To simplify notation we set

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}$$

and let A be the matrix $A = \begin{bmatrix} a_{jk} \end{bmatrix}$. Then the equations relating the two frames can be written in matrix form

$$\mathbf{e}' = A\mathbf{e}.$$

Given a connection ∇ we have

$$\nabla e_j = \sum_k \omega_{jk} e_k$$

where the ω_{jk} are ordinary (this is $\mathbb{R}$ valued) 1-forms. These are the ***connections forms*** determined by ∇ and the frame $\mathbf{e}$.

Proposition 5. *The connection forms ω_{jk} determine the connection on U. Conversely given any one forms ω_{jk} on the domain U of the frame $\mathbf{e}$, there is a unique connection ∇ that has these forms as its connection forms.*

Problem 4. Give a more precise statement of this result and prove it. $\square$

Let ω'_{jk} be the connection forms for the frame $\mathbf{e}'$. Then

$$\begin{aligned}
\sum_k \omega'_{jk} e'_k &= \nabla e'_j \\
&= \nabla \sum_\ell a_{j\ell} e_\ell \\
&= \sum_\ell da_{j\ell} e_\ell + \sum_\ell a_{j\ell} \nabla e_\ell \\
&= \sum_k da_{jk} e_k + \sum_\ell a_{j\ell} \sum_k \omega_{\ell k} e_k \\
&= \sum_k \Big(da_{jk} + \sum_\ell a_{j\ell} \omega_{\ell k} \Big) e_k.
\end{aligned}$$

I find this type of fiddling with indices annoying and hard to follow. We can do the same calculation in matrix form.

Problem 5. Show that the calculation just given in matrix notation is

$$\omega' \mathbf{e}' = \nabla \mathbf{e}' = \nabla(A\mathbf{e}) = dAe + A\nabla \mathbf{e} = (dA + A\omega)\mathbf{e}$$

$\square$

So to do the complete calculation relating ω and ω'

$$\begin{aligned}
\omega' \mathbf{e}' &= \nabla(A\mathbf{e}) \\
&= dA\mathbf{e} + A\nabla \mathbf{e} \\
&= dA\mathbf{e} + A\omega \mathbf{e} \\
&= (dA + A\omega)\, \mathbf{e} \\
&= (dA + A\omega)\, A^{-1} \mathbf{e}'
\end{aligned}$$

which gives the following equivalent formulas

$$\begin{aligned}
\omega' &= (dA + A\omega)\, A^{-1} \\
\omega' A &= dA + A\omega
\end{aligned}$$

Problem 6. We will be taking the derivative of the inverse function on matrices. Here is what is needs. Let $U \subseteq M$ be an open set and $A\colon U \to \mathrm{GL}(m,\mathbb{R})$ a smooth function. Show

$$dA^{-1} = -A^{-1}dAA^{-1}.$$

Hint: We have $AA^{-1} = I$ which is constant. Therefore by the product rule $0 = dAA^{-1} + AdA^{-1}$. □

We would now like to extend the definition of ∇ from a map $\nabla A^0(M,E) \to A^1(M,E)$ to a map $d^\nabla\colon A^k(M,E) \to A^{k+1}(M,E)$. If $\mathbf{e}$ is local frame for E over the open set $U \subseteq M$, then and $\alpha \in A^k(M,E)$ can be uniquely written as

$$\alpha = \sum_k \alpha_k e_k = \boldsymbol{\alpha}\mathbf{e}.$$

Then the natural definition of $d^\nabla\alpha$ is

$$\begin{aligned} d^\nabla\alpha &= d\boldsymbol{\alpha}\mathbf{e} + (-1)^k\boldsymbol{\alpha} \wedge \omega\mathbf{e} \\ &= \big(d\boldsymbol{\alpha} + (-1)^k\boldsymbol{\alpha} \wedge \omega\big)\mathbf{e} \end{aligned}$$

Proposition 6. *This definition is independent of the choice of local frame used to define dc.*

Problem 7. Prove this. *Hint:* Let $\mathbf{e}' = A\mathbf{e}$ be anther local frame and write

$$\alpha = \boldsymbol{\alpha}\mathbf{e} = \boldsymbol{\alpha}'\mathbf{e}'.$$

Show

$$\boldsymbol{\alpha}' = \boldsymbol{\alpha}A^{-1}$$

and therefore

$$\begin{aligned} d\boldsymbol{\alpha}' &= (d\boldsymbol{\alpha})A^{-1} + (-1)^k\boldsymbol{\alpha} \wedge dA^{-1} \\ &= (d\boldsymbol{\alpha})A^{-1} - (-1)^k\boldsymbol{\alpha} \wedge A^{-1}dAA^{-1}. \end{aligned}$$

Use this along with $\omega' = dAA^{-1} + A\omega A^{-1}$ to show

$$d\boldsymbol{\alpha}' + (-1)^k\boldsymbol{\alpha}' = \big(d\boldsymbol{\alpha} + (-1)^k\boldsymbol{\alpha} \wedge \omega\big)A^{-1}$$

and then use this to verify

$$\big(d\boldsymbol{\alpha} + (-1)^k\boldsymbol{\alpha} \wedge \omega\big)\mathbf{e} = \big(d\boldsymbol{\alpha}' + (-1)^k\boldsymbol{\alpha}' \wedge \omega'\big)\mathbf{e}'$$

□

Note that on $A^0(M,E)$ we have $d^\nabla = \nabla$. Since we have been generalizing the exterior derivative it would be reasonable to conjecture that $d^\nabla d^\nabla = 0$, but we will now see this is not the case. Let $s \in A^0(M,E)$. Then

$$\begin{aligned}
d^\nabla d^\nabla s &= d^\nabla(\omega \mathbf{e}) \\
&= d\omega\mathbf{e} - \omega \wedge d^\nabla \mathbf{e} \\
&= d\omega\mathbf{e} - \omega \wedge \omega\mathbf{e} \\
&= \big(d\omega - \omega \wedge \omega\big)\mathbf{e}
\end{aligned}$$

We give the 2 form that appear here a name. For a local frame $\mathbf{e}$ the form

$$\Omega = d\omega - \omega \wedge \omega$$

is the ***curvature form*** for the connection in this frame and the equation

$$d\omega = \omega \wedge \omega + \Omega$$

is the ***Cartan structural equation***.

Problem 8. Let $\alpha \in A^k(U,E)$ be given in terms of a local frame $\mathbf{e}$ on U by

$$\alpha = \boldsymbol{\alpha}\mathbf{e}.$$

Show

$$d^\nabla d^\nabla \alpha = \boldsymbol{\alpha} \wedge d^\nabla d^\nabla \mathbf{e} = \boldsymbol{\alpha} \wedge \Omega\mathbf{e}.$$

□

Theorem 7. *Let $\mathbf{e}$ and $\mathbf{e}'$ be two frames over the open set $U \subseteq M$ for the vector bundle $E \to M$ with $\mathbf{e}' = A\mathbf{e}$. Then the curvature forms Ω and Ω' are related by*

$$\Omega' = A\Omega A^{-1}.$$

Problem 9. Prove this. *Hint:* We know $\omega' = dAA^{-1} + A\omega A^{-1}$. Thus

$$\begin{aligned}
\Omega' &= d\omega' - \omega' \wedge \omega' \\
&= d(dAA^{-1} + A\omega A^{-1}) - (dAA^{-1} + A\omega A^{-1}) \wedge (dAA^{-1} + A\omega A^{-1}) \\
&\quad \vdots
\end{aligned}$$

A tedious calculation using $dA^{-1} = -A^{-1}dAA^{-1}$ etc.

$$\begin{aligned}
&\quad \vdots \\
&= A(d\omega - \omega \wedge \omega)A^{-1}
\end{aligned}$$

as required. □

It is now easy to compute $d\Omega$.

Theorem 8 (The Bianchi identity)**.** *The exterior derivative of Ω is*

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega.$$

Proof. We know $d\omega = \omega \wedge \omega + \Omega$ and $\Omega = d\omega - \omega \wedge \omega$. Thus

$$\begin{aligned} d\Omega &= d(d\omega - \omega \wedge \omega) \\ &= -d\omega \wedge \omega + \omega \wedge d\omega \\ &= -(\omega \wedge \omega + \Omega) \wedge \omega + \omega \wedge (\omega \wedge \omega + \Omega) \\ &= \omega \wedge \Omega - \Omega \wedge \omega. \end{aligned}$$

□

The following will be used later and is good practice in working with this set of ideas.

Lemma 9 (Generalized Bianchi identity)**.** *Let Ω the curvature forms of a connection and k a positive integer. Then*

$$d\Omega^k = \omega \wedge \Omega^k - \Omega^k \wedge \omega.$$

Problem 10*.* Prove this. *Hint:* Here is the calculation when $k = 3$.

$$\begin{aligned} d\Omega^3 &= d\Omega \wedge \Omega^2 + \Omega \wedge d\Omega \wedge \Omega + \Omega^2 \wedge d\Omega \\ &= (\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega^2 + \Omega \wedge (\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega \\ &\qquad + \Omega^2 \wedge (\omega \wedge \Omega - \Omega \wedge \omega) \\ &= \omega \wedge \Omega^3 - \Omega \wedge \omega \wedge \Omega^2 + \Omega \wedge \omega \wedge \Omega^2 - \Omega^2 \wedge \omega \wedge \Omega \\ &\qquad + \Omega^2 \wedge \omega \wedge \Omega - \Omega^3 \wedge \omega \\ &= \omega \wedge \Omega^k - \Omega^k \wedge \omega. \end{aligned}$$

Show that this telescoping pattern works for all k. Or if you want to be a bit more elegant, the proof by induction on k is shorter. □

We are now in a position to define the first Chern class defined by the connection ∇. Recall that for any square matrices A and B of the same size that

$$\operatorname{tr}(AB) = \operatorname{tr}(BA).$$

Therefore $\operatorname{tr}(AB - BA) = 0$. In particular this implies that if Ω and Ω' are the curvature forms defined by two local frames $\mathbf{e}$ and $\mathbf{e}'$, then

$$\operatorname{tr}(\Omega') = \operatorname{tr}(A\Omega A^{-1}) = \operatorname{tr}(\Omega A^{-1} A) = \operatorname{tr}(\Omega).$$

Therefore the 2-form $\operatorname{tr}(\Omega)$ is independent of the frame used to find Ω and thus $\operatorname{tr}(\Omega)$ is a globally defined 2-form on M, depending only on the connection ∇.

Proposition 10. *The form* $\operatorname{tr}(\Omega)$ *is closed. That is*

$$d\left(\operatorname{tr}(\Omega)\right) = 0.$$

Problem 11. Prove this by explaining why the following calculation makes sense.

$$\begin{aligned} d\left(\operatorname{tr}(\Omega)\right) &= \operatorname{tr}(d\Omega) \\ &= \operatorname{tr}(\omega \wedge \Omega - \Omega \wedge \omega) \\ &= 0 \end{aligned}$$

□

This shows that $\operatorname{tr}(\Omega)$ represents a de Rham cohomology class $[\operatorname{tr}(\Omega)] \in H^2_{\mathrm{dR}}(M)$. Our next goal is to show that in some cases this cohomology class is independent of the connection. Let ∇_0 and ∇_1 be connections on $E \to M$. Let for $t \in [0,1]$ let ∇_t be the connection

$$\nabla_t = (1-t)\nabla_0 + t\nabla_1$$

Then

$$\nabla_t \mathbf{e} = (1-t)\nabla_0\mathbf{e} + t\nabla_1\mathbf{e} = (1-t)\omega_0\mathbf{e} + t\omega_1\mathbf{e} = (\omega_0 + t\eta)\mathbf{e}$$

where

$$\eta = \omega_1 - \omega_0.$$

The matrix of 1-forms transforms nicely under a change of frame $\mathbf{e}' = A\mathbf{e}$:

$$\eta' = \omega_1' - \omega_0' = (dAA^{-1} + A\omega_1A^{-1}) - (dAA^{-1} + A\omega_0A^{-1}) = A\eta A^{-1}.$$

This implies that the 1-form

$$\alpha := \operatorname{tr}(\eta)$$

is globally defined on M.

Lemma 11. *Let P and Q be square matrices of the same size with elements in a ring R such that for all elements of P anti-commute with elements of Q. That is $p_{ij}q_{k\ell} = -q_{k\ell}p_{ij}$ for all i, j, k, ℓ. Then*

$$\operatorname{tr}(PQ + QP) = 0$$

Problem 12. Prove this. *Hint:* A straightforward calculation shows that $\operatorname{tr}(PQ) = \sum_{i,j} p_{ij}q_{jk}$ and $\operatorname{tr}(QP) = \sum_{i,j} q_{ij}p_{ji}$. Now just do a change of indices in one of the sums. □

Using this lemma and that $\omega_t = \omega_0 + t\eta$ we have

$$\begin{aligned} d\alpha &= d\operatorname{tr}(\eta) \\ &= \operatorname{tr}(d\eta - \eta \wedge \omega_t - \omega_t \wedge \eta) \\ &= \operatorname{tr}(d\eta - \eta \wedge (\omega_0 + t\eta) - (\omega_0 + t\eta) \wedge \eta) \\ &= \operatorname{tr}(d\eta - \eta \wedge \omega_0 - \omega_0 \wedge \eta - 2t\eta \wedge \eta) \end{aligned}$$

The curvature form of ∇_t is

$$\begin{aligned}\Omega_t &= d\omega_t - \omega_t \wedge \omega_t \\ &= d(\omega_0 + t\eta) - (\omega_0 + t\eta) \wedge (\omega_0 + t\eta) \\ &= d\omega_0 - \omega_0 \wedge \omega_0 - t((d\eta - \eta \wedge \omega_0 - \omega_0 \wedge \eta) - t^2 \eta \wedge \eta\end{aligned}$$

Thus

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}(\Omega_t) &= \frac{d}{dt}\operatorname{tr}\big(d\omega_0 - \omega_0 \wedge \omega_0 - t((d\eta - \eta \wedge \omega_0 - \omega_0 \wedge \eta) - t^2 \eta \wedge \eta\big) \\ &= \operatorname{tr}(d\eta - \eta \wedge \omega_0 - \omega_0 \wedge \eta - 2t\eta \wedge \eta) \\ &= d\alpha\end{aligned}$$

and therefore

$$\begin{aligned}\operatorname{tr}(\Omega_1) - \operatorname{tr}(\Omega_0) &= \int_0^1 \frac{d}{dt}\operatorname{tr}(\Omega_t)\,dt \\ &= \int_0^1 d\alpha \\ &= d\alpha.\end{aligned}$$

To summarize the latest subplot:

Proposition 12. *Given any two connections, ∇_0 and ∇_1, on the vector bundle $E \to M$ with curvature forms Ω_0 and Ω_1 that both the 2-forms $\operatorname{tr}(\Omega_0)$ and Ω_1 are closed and there is a 1-form α with*

$$\operatorname{tr}(\Omega_1) - \operatorname{tr}(\Omega_0) = d\alpha.$$

Therefore for any connection ∇ on $E \to M$, the de Rham cohomology class $[\operatorname{tr}(\Omega)] \in H^1_{\mathrm{dR}}(M)$ only depends on the vector bundle and not on the connection ∇. ☐

We would like to define higher degree versions of this form. Our (really Chern's) generalization will start with replacing $\operatorname{tr}(\Omega)$ with $\operatorname{tr}(\Omega^k)$ where Ω is the curvature form of a connection and k is a positive integer.

Lemma 13. *Let Ω be the curvature form of a connection on the vector bundle $E \to M$. Then for any positive integer k the forms $\operatorname{tr}(\Omega^k)$ is closed.*

Problem 13. Prove this. *Hint:* Here is a calculation that does trick for $k = 3$. It is based on the Bianchi identity $d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$ and that for any matrix of form β that $\operatorname{tr}(\beta \wedge \Omega) = \operatorname{tr}(\Omega \wedge \beta)$.

$$\begin{aligned}
d\operatorname{tr}(\Omega^3) &= \operatorname{tr}\left(d(\Omega)^3\right) \\
&= \operatorname{tr}(d\Omega \wedge \Omega^2) + \operatorname{tr}(\Omega \wedge d\Omega \wedge \Omega) + \operatorname{tr}(\Omega^2 \wedge d\Omega) \\
&= 3\operatorname{tr}(d\Omega \wedge \Omega^2) \\
&= 3\operatorname{tr}\big((\omega \wedge \Omega - \Omega \wedge \omega) \wedge \Omega^2\big) \\
&= 3\operatorname{tr}(\omega \wedge \Omega^2) - 3\operatorname{tr}(\Omega \wedge \omega \wedge \Omega) \\
&= 3\operatorname{tr}(\omega \wedge \Omega^2) - 3\operatorname{tr}(\omega \wedge \Omega^2) \\
&= 0
\end{aligned}$$

□

Lemma 14. *It ∇_1 and ∇_0 are connections on the vector bundle $E \to M$ with curvature forms Ω_0 and Ω_1 and k is a positive integer, then is a $(2k-1)$-form such that*

$$\operatorname{tr}(\Omega_1^k) - \operatorname{tr}(\Omega_0^k) = d\alpha.$$

Problem 14. Prove this. *Hint:* Here is the calculation when $k = 3$. As before let $\nabla_t = (1-t)\nabla_1 + t\nabla_1$. Again let

$$\eta = \omega_1 - \omega_0.$$

Under a change of frame $\mathbf{e}' = A\mathbf{e}$ we have that the curvature forms Ω_t and the form η transform by

$$\eta' = A\eta A^{-1}, \qquad \Omega_t' = A\Omega_t A^{-1}$$

and therefore the form

$$\beta_t = \operatorname{tr}(\eta \wedge \Omega_t^2)$$

is defined independently of the fame $\mathbf{e}$ and thus β_t is globally defined on M. By the generalized Bianchi identity

$$\begin{aligned}
d\beta_t &= d\operatorname{tr}(\eta \wedge \Omega_t^2) \\
&= \operatorname{tr}(d\eta \wedge \Omega_t^2) - \operatorname{tr}(\eta \wedge d\Omega_t^2) \\
&= \operatorname{tr}(d\eta \wedge \Omega_t^2) - \operatorname{tr}(\eta \wedge (\omega_t \wedge \Omega_t^2 - \Omega_t^2 \wedge \omega_t)) \\
&= \operatorname{tr}(d\eta \wedge \Omega_t^2) - \operatorname{tr}(\eta \wedge \omega_t \wedge \Omega_t^2) + \operatorname{tr}(\eta \wedge \Omega_t^2 \wedge \omega_t) \\
&= \operatorname{tr}(d\eta \wedge \Omega_t^2) - \operatorname{tr}(\eta \wedge \omega_t \wedge \Omega_t^2) - \operatorname{tr}(\omega_t \wedge \eta \wedge \Omega_t^2) \\
&= \operatorname{tr}\big((d\eta - \eta \wedge \omega_t - \omega_t \wedge \eta) \wedge \Omega_t^2\big)
\end{aligned}$$

(You should check that what looks like a sign error (i.e $\operatorname{tr}(\eta \wedge \Omega_t^2 \wedge \omega_t) = -\operatorname{tr}(\omega_t \wedge \eta \wedge \Omega_t^2)$) is really an application of Lemma 11.) Now note

$$\frac{d}{dt}\omega_t = \frac{d}{dt}\big((1-t)\omega_0 + t\omega_1\big) = \omega_1 - \omega_0 = \eta$$

and therefore

$$\begin{aligned}\frac{d}{dt}\Omega_t &= \frac{d}{dt}\big(d\omega_t - \omega_t \wedge \omega_t\big)\\ &= d\eta - \eta \wedge \omega_t - \omega_t \wedge \eta.\end{aligned}$$

Using this and the formula for $d\beta_t$ we find

$$\begin{aligned}\frac{d}{dt}\operatorname{tr}(\Omega_t^3) &= \operatorname{tr}\left(\left(\frac{d}{dt}\Omega_t\right) \wedge \Omega_t^2\right) + \operatorname{tr}\left(\Omega_t \wedge \left(\frac{d}{dt}\Omega_t\right) \wedge \Omega_t\right)\\ &\qquad + \operatorname{tr}\left(\Omega_t^2 \wedge \left(\frac{d}{dt}\Omega_t\right)\right)\\ &= 3\operatorname{tr}\left(\left(\frac{d}{dt}\Omega_t\right) \wedge \Omega_t^2\right)\\ &= 3\operatorname{tr}\big((d\eta - \eta \wedge \omega_t - \omega_t \wedge \eta) \wedge \Omega_t^2\big)\\ &= 3\, d\beta_t.\end{aligned}$$

This can be integrated to give

$$\begin{aligned}\operatorname{tr}(\Omega_1^3) - \operatorname{tr}(\Omega_0^3) &= \int_0^1 \frac{d}{dt}\operatorname{tr}(\Omega_t^3)\, dt\\ &= \int_0^1 3\, d\beta_t\, dt\\ &= 3\, d\int_0^1 \beta_t\, dt\\ &= d\alpha\end{aligned}$$

where

$$\alpha = 3\int_0^1 \beta_t\, dt = 3\int_0^1 \operatorname{tr}(\eta \wedge \Omega_t^2)\, dt. \qquad \square$$

All of this works for either real or complex vector bundles. In the case of complex vector bundles we choose the frames

$$\mathbf{e} = \begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_m \end{bmatrix}$$

to be complex frames (so that for each x in the domain of the fame the vectors $e_1(x), \ldots, e_m(x)$ are a basis for E_x over the complex numbers) and the connection forms ω_{jk} and curvature forms Ω_{jk} will complex rather than real valued.

Remark 15. For real vector bundles there is a complication. If $E \to M$ is a smooth real vector bundle over a smooth manifold, then (and I do not know an elementary proof of this) there is smooth choice of inner product

$\langle\, , \rangle$ on the fibers of E and a connection ∇ that is compatible with this inner product in the sense that if s_1 and s_2 are sections of E than

$$X\langle s_1, s_2\rangle = \langle \nabla_X s_1, s_2\rangle + \langle s_1, \nabla_X s_2\rangle.$$

Anther way to say this is that $\mathbf{e}$ is a local frame that is pointwise an orthogonal basis of E_x, then the connection forms ω_{jk} and the curvature forms Ω_{jk} of ∇ are skew symmetric functions of the indices j and k. In particular this implies that $\Omega_{jj} = 0$ for all j. Thus

$$\operatorname{tr}(\Omega) = 0$$

and so the cohomology class $[\operatorname{tr}(\Omega)]$ is zero, which is not very interesting. More generally the skew symmetry implies that for all odd positive integers k that

$$\operatorname{tr}(\Omega^k) = 0.$$

But on the positive side when k is even $\operatorname{tr}(\Omega^k)$ need not vanish and the cohomology classes $[\operatorname{tr}(\Omega^k)] \in H^k_{\mathrm{dR}}(M)$ are non-trivial topologically invariants of the vector bundle $E \to M$. $\square$

For complex bundles there is no problem with the forms $\operatorname{tr}(\Omega^k)$ vanishing for odd k. In terms of geometry the forms $\operatorname{tr}(\Omega)$ are not the most natural choice. Let $E \to M$ be a complex vector bundle over M with connection ∇ and let ω and Ω be the connection and curvature forms of ∇ relative to a complex frame $\mathbf{e}$. Define forms $c_k(E, \nabla)$ by

$$\det\left(I + t\frac{i}{2\pi}\Omega\right) = 1 + tc_1(E,\nabla) + t^2 c_2(E,\nabla) + \cdots + t^m c_m(E,\nabla)$$

where m is the fiber dimension of E and i, as usual, is $i = \sqrt{-1}$.

Problem 15. Show that the definition of the forms $c_k(E,\nabla)$ is independent change of frame $\mathbf{e}' = A\mathbf{e}$ and therefore the forms $c_k(E,\nabla)$ are globally defined on M. $\square$

Proposition 16 (Newton's identities)**.** *Let $x_1, x_2, \ldots, x_m$ be elements of any commutative ring with identity. Define two sets of polynomials in these elements. First the* ***elementary symmetric functions****, $\sigma_0 = 1$, σ_1 ,$\ldots$,σ_m by in these elements are defined by*

$$(1 + tx_1)(1 + tx_2)\cdots(1 + tx_m) = 1 + \sigma_1 t + \sigma_2 t^2 + \cdots + t^m \sigma_m$$

with the convention that $\sigma_k = 0$ for $k > m$. Define $s_0 = m$, s_1, s_2, $\ldots$ by

$$s_k = x_1^k + x_2^k + \cdots + x_m^k.$$

Then

$$k\sigma_k = \sum_{j=1}^{k} (-1)^{j-1}\sigma_{k-j} s_j.$$

Problem 16. Prove this. *Hint:* Consider the case where the ring is $R = \mathbb{Z}[x_1, \ldots, x_m]$ and the elements $x_1, \ldots, x_m$ are variables. If the result holds on this case, then in holds over all commutative rings with unit by specialization. Let $f(t) \in R[[t]]$ (the ring of formal power series over R) by

$$f(t) = (1 + tx_1)(1 + tx_2) \cdots (1 + tx_m) = 1 + \sigma_1 t + \sigma_2 t^2 + \cdots + t^m \sigma_m.$$

Then

$$\begin{aligned} \frac{f'(t)}{f(t)} &= \sum_{j=1}^{m} \frac{x_j}{1 + tx_j} \\ &= \sum_{j=1}^{m} \sum_{k=0}^{\infty} (-1)^k x_j^{k+1} t^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{k=0}^{\infty} x_j^{k+1} \right) (-1)^k t^k \\ &= \sum_{k=0}^{\infty} \left(\sum_{j=1}^{m} x_j^{k+1} \right) (-1)^k t^k \\ &= \sum_{k=0}^{\infty} (-1)^k s_{k+1} t^k. \end{aligned}$$

Therefore

$$f'(t) = f(t) \sum_{k=0}^{\infty} (-1)^k s_{k+1} t^k.$$

Compare the coefficients of t in this to complete the proof. $\square$

Proposition 17. *Let A be an $m \times m$ matrix with elements in a communicative ring with unit R. Define two sets of polynomials the elements of A by*

$$\det(I + tA) = \sigma_0(A) + t\sigma_1(A) + \cdots + t^m \sigma_m(A).$$

and

$$s_k(A) = \operatorname{tr}(A^k)$$

with the convention that $s_0(A) = m$. Then

$$k\sigma_k(A) = \sum_{j=1}^{k} (-1)^{j-1} \sigma_{k-j}(A) s_j(A).$$

Problem 17. Prove this. $\square$

Theorem 18. *Let $E \to M$ be a complex vector bundle over a smooth manifold M with a connection ∇. Then the Chern forms*

$$c_k(E, \nabla)$$

are closed and the cohomology class $[c_k(E,\nabla)]$ is independent of the connection. Thus the ***Chern class***

$$c_k(E) = [c_k(E,\nabla)]$$

is an topological invariant of the vector bundle $E \to M$.

Problem 18. Prove this. *Hint:* One way is to use induction on k. For $k=1$ we have

$$c_1(E,\nabla) = \frac{i}{2\pi}\operatorname{tr}(\Omega)$$

where Ω is the curvature form of ∇ relative to a complex frame. We have already seen that this is closed and the resulting cohomology class is independent of the connection.

We now do the induction step from $k=2$ to $k=3$ and leave the general case to you. From Proposition 17 we have

$$\begin{aligned} 3c_3(E,\nabla) &= \left(\frac{i}{2\pi}\right)^3 \sigma_3(\Omega) \\ &= \left(\frac{i}{2\pi}\right)^3 \big(\sigma_2 \wedge \operatorname{tr}(\Omega) - \sigma_1(\Omega)\wedge \operatorname{tr}(\Omega^2)\big) \\ &= \left(\frac{i}{2\pi}\right) c_2(E,\nabla)\wedge \operatorname{tr}(\Omega) - \left(\frac{i}{2\pi}\right)^2 c_1(E,\nabla)\wedge \operatorname{tr}(\Omega^2). \end{aligned}$$

By the induction hypothesis this a product of closed forms and thus it is closed.

If ∇_0 and ∇_1 are both connections on E then we have already seen that

$$[c_1(E,\nabla_1)] = \frac{i}{2\pi}[\operatorname{tr}(\Omega_1)] = \frac{i}{2\pi}[\operatorname{tr}(\Omega_0) = [c_1(E,\nabla_0)]$$

where $[\alpha]$ is the de Rham cohomology class of the closed form α. We again use induction on k. And again here is the step from $k=2$ to $k=3$ with the general case left to you. Using the induction hypothesis and that by Lemma 14 we have $[\operatorname{tr}(\Omega_1^j)] = [\operatorname{tr}(\Omega_0^j)]$ for all positive integers j. Therefore

$$\begin{aligned} 3[c_3(E,\nabla_1)] &= \left(\frac{i}{2\pi}\right)[c_2(E,\nabla_1)]\wedge[\operatorname{tr}(\Omega_1)] - \left(\frac{i}{2\pi}\right)^2 [c_1(E,\nabla_1)]\wedge[\operatorname{tr}(\Omega_1^2)] \\ &= \left(\frac{i}{2\pi}\right)[c_2(E,\nabla_0)]\wedge[\operatorname{tr}(\Omega_1)] - \left(\frac{i}{2\pi}\right)^2 [c_1(E,\nabla_0)]\wedge[\operatorname{tr}(\Omega_1^2)] \\ &= 3[c_3(E,\nabla_0)]. \end{aligned}$$

□

Let $E \to M$ be a complex vector bundle over the smooth manifold M. Then a ***Hermitian metric*** on E is a choice of Hermitian inner product $\langle\,,\rangle_x$ on each fiber E_x such that if s_1 and s_2 are sections of E, then the function $x \mapsto \langle s_1(x), s_2(x)\rangle_x$ is smooth.

Proposition 19. *Every smooth complex vector bundle over a smooth manifold has a Hermitian metric.*

Proof. Prove this. *Hint:* Locally this is easy. Let $e_1, \ldots, e_m$ be a local frame for E over the open set U. Then let $\langle\,,\rangle_x$ be the unique Hermitian inner product on E_x that makes $e_1(x), e_2(x), \ldots, e_m(x)$ into a Unitary (the complex version of orthonormal) basis of E_x. Now use a partition of unity argument to get a global metric on E. □ □

Let ∇ be a connection on $E \to M$ and assume that E has a Hermitian metric $\langle\,,\rangle$. Then the connection is a ***metric connection*** if and only for all smooth sections s_1 and s_2 of E that the following version of the product rule holds

$$d\langle s_1, s_2\rangle = \langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle$$

holds.

Proposition 20. *Let $E \to M$ be a complex vector bundle with Hermitian metric $\langle\,,\rangle$. Then there is a metric connection for $\langle\,,\rangle$.*

Proof. Let $U \subseteq M$ be an open subset of M and assume that $\nabla_1, \ldots, \nabla_r$ are metric connections for the restriction of E to U. Let $\rho_1, \ldots, \rho_r$ be smooth functions on M such that $\sum_k \rho_k = 1$. Then $\nabla = \sum_k \rho_k \nabla_k$ is a metric connection on the restriction of E to U. To see this note that as each ∇_k is metric we have

$$\langle \nabla_k s_1, s_2\rangle + \langle s_2, \nabla_k s_2\rangle = d\langle s_1, s_2\rangle$$

for any smooth sections s_1 and s_2. Therefore

$$\begin{aligned}
\langle \nabla s_1, s_2\rangle + \langle s_1, \nabla s_2\rangle &= \sum_k \langle \rho_k \nabla_k s_1, s_2\rangle + \sum_k \langle s_2, \rho_k \nabla_k s_2\rangle \\
&= \sum_k \rho_k \langle \nabla_k s_1, s_2\rangle + \sum_k \rho_k \langle s_2, \nabla_k s_2\rangle \\
&= \sum_k \rho_k \big(\rho_k \langle \nabla_k s_1, s_2\rangle + \langle s_2, \nabla_k s_2\rangle\big) \\
&= \sum_k \rho_k \, d\langle s_1, s_2\rangle \\
&= d\langle s_1, s_2\rangle.
\end{aligned}$$

The rest of the proof is a standard partition of unity argument. □

We now look at the special case of complex line bundles. Let $L \to M$ be a complex line bundle over M. We assume that L has a Hermitian metric $\langle\,,\rangle$ and a metric connection ∇. Let $\mathbf{e}$ be a unitary frame for L. Let ω be the connection form for ∇ and this frame. Then

$$\nabla \mathbf{e} = \omega \mathbf{e}$$

where, as L is a line bundle, ω is a 1×1 matrix, that is just a $\mathbb{C}$ valued 1-form. As $\langle \mathbf{e},\mathbf{e}\rangle \equiv 1$ we have

$$\begin{aligned}
0 &= d1 \\
&= d\langle \mathbf{e},\mathbf{e}\rangle \\
&= \langle \nabla\mathbf{e},\mathbf{e}\rangle + \langle \mathbf{e},\nabla\mathbf{e}\rangle \\
&= \langle \omega\mathbf{e}\mathbf{e}\rangle + \langle \mathbf{e},\omega\mathbf{e}\rangle \\
&= (\omega+\overline{\omega})\langle \mathbf{e}\mathbf{e}\rangle \\
&= \omega+\overline{\omega}
\end{aligned}$$

and therefore

$$\overline{\omega} = -\omega.$$

Thus ω takes its values in the pure imaginary numbers. Also as ω is 1-from we have we have that $\omega\wedge\omega = 0$ and thus the curvature form in this frame is

$$\Omega = d\omega - \omega\wedge\omega = d\omega.$$

If e' is anther unitary frame for L, as both $\mathbf{e}$ and $\mathbf{e}'$ have length 1 we have

$$\mathbf{e}' = e^{i\theta}\mathbf{e}$$

for some smooth real valued function θ. It follows that

$$\omega' = \omega + i\,d\theta$$

where ω' is the connection form for ∇ and $\mathbf{e}'$.

Problem 19. Verify the last equation and convince yourself that θ is just the angle between $\mathbf{e}$ and $\mathbf{e}'$. $\square$

Proposition 21. *Let $L\to M$ be a complex line bundle over a two dimensional oriented manifold M. Let $U\subseteq M$ be an open subset of M with smooth boundary and let*

We now assume that M is a oriented two dimensional manifold and that $L\to M$ is a complex line bundle over M. Let s be a section of L such that $s\neq 0$ in some punctured neighborhood of p_0. At p_0 the section s may vanish or be undefined. By choosing a coordinate system centered at p_0 and a local trivialization of L near p_0 we can assume think of s as a function from a neighborhood of 0 in $\mathbb{C}$ to $\mathbb{C}$ with $s(0)=0$. That is s can be represented geometrically as a vector field on the plane vanishing at the origin. An example of this set up is given in Figure 1. Now let

$$\mathbf{e}' = \frac{s}{\|s\|}$$

be s normalized to be zero at points where s and defined an nonvanishing. Let

$$\mathbf{e} = \text{any unit length section of } L \text{ defined near } p_0.$$

In particular $\mathbf{e}$ is defined at p_0. As before

$$\mathbf{e}' = e^{i\theta}\mathbf{e}.$$

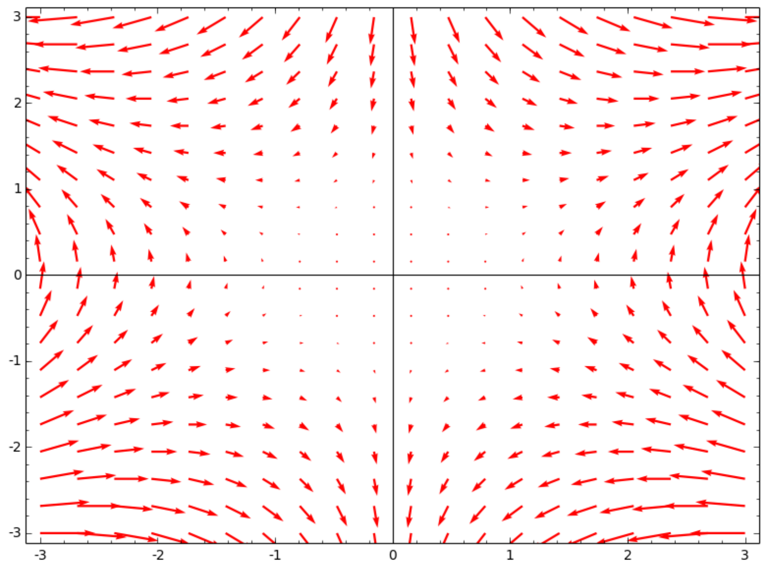

FIGURE 1. A section of a line bundle over a surface with an isolated zero at the origin (which corresponds to the point x_0 of M.)

Draw a simple closed curve γ around p_0 which is positively oriented with respect to the orientation of M. The set up now is represented by the picture in Figure 2.

Proposition 22. *Let M be a 2-dimensional oriented manifold and let $L \to M$ be a complex line bundle over M and assume that L has a Hermitian fiber metric. Let s be a section of L with defined and nonvanishing in a punctured neighborhood of $x_0 \in M$. (The section may may vanish at x_0.) Let $\mathbf{e}$ be a local unit length section of L defined in a neighborhood of x_0. Set $\mathbf{e}' = \|s\|^{-1}s$. Also choose a smooth simple closed curve γ in M that bounds a disk in M containing x_0. (This setup is pictured in Figure 2.) Then the **winding number**, $I(x_0, s)$, of $\mathbf{e}'$ with respect to $\mathbf{e}$ is well defined and is the number of times that $\mathbf{e}'$ rotates around $\mathbf{e}$ as a point on γ moves once around γ once in the positive direction. (This is illustrated in Figure 3). The number $I(x_0, s)$ is an integer and is independent of the choice of the fiber metric on L, the choice of the nonvanishing section $\mathbf{e}'$ and the choice of the curve γ. Therefore, as the notation indicates, $I(s, x_0)$ only depends on the section s and the point x_0. Finally if M has a complex structure and ω is the connection form of of the Chern connection with respect to $\mathbf{e}$ and ω' is the connection form with respect to $\mathbf{e}'$ then, as we have seen above,*

$$\omega' = \omega + i\,d\theta$$

for a smooth function θ (which can be thought of as the angle between $\mathbf{e}$ and $\mathbf{e}'$). Then

$$\int_\gamma \omega' - \int_\gamma \omega = i \int_\gamma d\theta = 2\pi i I(s, x_0).$$

Problem 20. As good exercise in reviewing your algebraic topology class prove this. □

Theorem 23. *Let $L \to M$ be a complex line bundle over a compact oriented 2-dimensional surface. Let $x_0, x_1, \ldots, x_m$ be a finite collection of points on M and let s be a nonvanishing section of L on $M \smallsetminus \{x_1, \ldots, x_m\}$. Let Ω be the curvature for of some connection on L. Then*

$$\int_M \Omega = 2\pi i \sum_{j=1}^{m} I(s, x_j).$$

Problem 21. Prove this. *Hint:* Since $\int_M \Omega$ is independent of the connection (as the cohomology class $[\Omega] \in H^2_{\mathrm{dR}}(M)$ is independent of the connection)

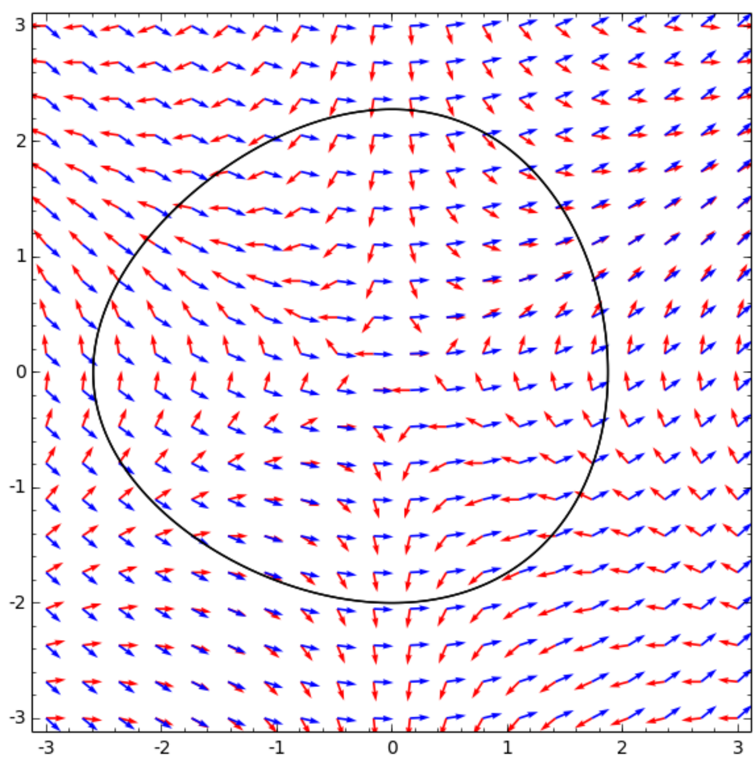

FIGURE 2. The red is the section $\mathbf{e}' = \|s\|^{-1}s$. The blue is the nonvanishing section $\mathbf{e}$. And we have the simple closed curve γ which encloses the point p_0 where s is zero or undefined.

we can put a fiber metric on L and assume that the connection ∇ is the Chern connection. Around each point x_j choose a small disk D_j with smooth boundary. By making the disks smaller we can assume that $D_j \cap D_k = \varnothing$ for $j \neq k$ and that we can choose a unit length section, $\mathbf{e}$, of L restricted to $\bigcup_{j=1}^m D_j$. Let ω be the connection form of ∇ with respect to the section $\mathbf{e}$. Let $\mathbf{e}' = \|s\|^{-1}s$. This is a unit length section of L over $M \smallsetminus \{x_1, \ldots, x_m\}$. Let ω' be the connection form of ∇ with respect to $\mathbf{e}'$. Then the curvature form Ω can be expressed in terms of either ω or ω' as

$$\Omega = d\omega = d\omega'.$$

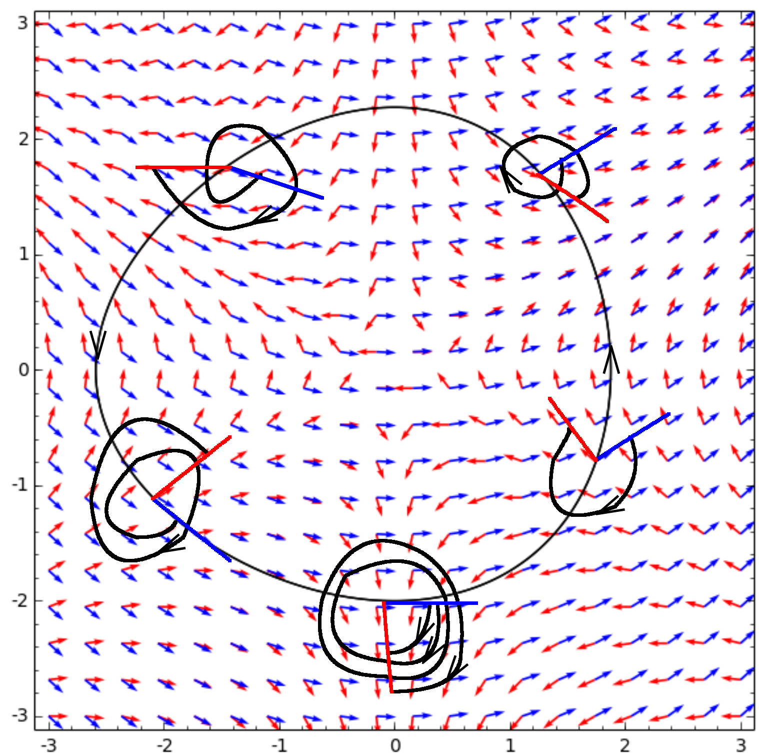

Figure 3. The red is the section $\mathbf{e}' = \|s\|^{-1}s$. The blue is the nonvanishing unit length section $\mathbf{e}$. And we have the simple closed curve γ which encloses the point p_0 where s is zero or undefined. In this figure the vector field $\mathbf{e}'$ rotates twice in the negative direction with respect to the vector field $\mathbf{e}$ as a point makes one circuit around the curve γ. Therefore $I(s, x_0) = -2$

Now explain why the following calculation works:

$$\begin{aligned}
\int_M \Omega &= \int_{M \setminus \bigcup_{j=1}^m D_j} \Omega + \int_{\bigcup_{j=1}^m D_j} \Omega \\
&= \int_{M \setminus \bigcup_{j=1}^m D_j} d\omega' + \int_{\bigcup_{j=1}^m D_j} d\omega \\
&= -\sum_{j=1}^m \int_{\partial D_j} \omega' + \sum_{j=1}^m \int_{\partial D_j} \omega \\
&= \sum_{j=1}^m \int_{\partial D_j} (\omega - \omega') \\
&= -2\pi i \sum_{j=1}^m I(s, x_j).
\end{aligned}$$

This calculation uses Stokes' Theorem for surfaces with boundary and the tricky part is keeping track of orientations and signs. □