An example of a torus bundle.

Here are more details on the example of Problem 5 from the homework. Let

$$H = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

be the upper half plane in $\mathbb{C}$. For each $z \in H$ we form the lattice

$$\mathbb{Z} \oplus \mathbb{Z}z := \{m + nz : m, n \in \mathbb{Z}\}.$$

As $\mathrm{Im}(z) > 0$ the complex numbers 1 and z are linearly independent over $\mathbb{R}$. Let E_z be the quotient

$$E_z := \mathbb{C}/(\mathbb{Z} \oplus \mathbb{Z}z).$$

This is torus. We now find isomorphisms between these tori. Use 1, i as a basis of $\mathbb{C}$ over $\mathbb{R}$. For $z = x + yi \in H$ let A_z be the matrix

$$A_z = \begin{bmatrix} 1 & -x/y \\ 0 & 1/y \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & -x/y \\ 0 & 1/y \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & -x/y \\ 0 & 1/y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

That is

$$A_z 1 = 1, \qquad A_z z = i.$$

Problem 1. If $w \in \mathbb{C}$, let $[w]_{E_z}$ be the equivalence class of w in the quotient $E_z = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}z)$. The show that the map $\psi_z \colon E_z \to E_i$ given by

$$\psi_z[w]_{E_z} = [A_z w]_{E_i}$$

is a group isomorphism and topologically is a homeomorphism. □

On the space $H \times \mathbb{C}$ define an equivalence relation by

$$(z_1, w_1) \sim (z_2, w_2) \qquad \Longleftrightarrow \qquad z_1 = z_2 \text{ and } w_2 - w_1 \in \mathbb{Z} \oplus \mathbb{Z}z.$$

Let $E = (H \times \mathbb{C})/\sim$ be the quotient space and let $p \colon E \to H$ be the map $p([z, w]) = z$ (where $[z, w]$ is the equivalence class of (z, w)).

Problem 2. With this set up let $p \colon E \to H$ is a torus bundle over H. It the trivial bundle in the sense that it is isomorphic to a product. *Hint:* Define $\Psi \colon E \to B \times E_i$ by

$$\Psi([z, w]) = (z, [A_z w]_{E_i}).$$

□