Mathematics 172 Homework, February 21, 2019.

Recall the plot to date on our study of age structured population growth. If we have three stages we let

$$\vec{N}(t) = \begin{bmatrix} n_1(t) & n_2(t) & n_3(t) \end{bmatrix}$$

where $n_j(t)$ is the number of individuals in stage j in year t. We have the ***Leslie matrix***

$$L = \begin{bmatrix} f_1 & f_2 & f_3 \\ p_1 & 0 & 0 \\ 0 & p_2 & 0 \end{bmatrix}.$$

In entries have the meanings

$$\begin{aligned} f_j &= \text{Fecundity of stage } j \text{ individuals} \\ &= \text{average number of offspring to a stage } j \text{ individual.} \\ p_j &= \text{Proportion of stage } j \text{ individuals that live to stage } j+1. \end{aligned}$$

The Leslie matrix tells us how the population changes from one year to the next:

$$\vec{N}(t+1) = L\vec{N}(t).$$

Thus if we know $\vec{N}(0)$ we can compute the numbers in the future by

$$\vec{N}(t) = L^t\vec{N}(0).$$

At least for matrices that are not too large this is easy to do on our calculators.

One of the pieces of information that is ***age distribution*** in year t. That is the proportion of the population that is in each stage. If

$$\begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix}$$

then the total number of individuals is

$$\text{tot}(t) = n_1(t) + n_2(t) + n_3(t)$$

and the age distribution is given by the vector

$$\frac{1}{\text{tot}(t)}\vec{N}(t) = \frac{1}{\text{tot(t)}} \begin{bmatrix} n_1(t) \\ n_2(t) \\ n_3(t) \end{bmatrix} = \begin{bmatrix} \dfrac{n_1(t)}{\text{tot}(t)} \\ \\ \dfrac{n_2(t)}{\text{tot}(t)} \\ \\ \dfrac{n_3(t)}{\text{tot}(t)} \end{bmatrix}$$

We computed many examples of this on Quiz 13.

We have a ***stable age distribution*** if the age distribution stays the same from year to year. That this means is that for years t and $t+1$ we have

$$\frac{1}{\operatorname{tot}(t)}\vec{N}(t) = \frac{1}{\operatorname{tot}(t+1)}\vec{N}(t+1)$$

which can be rewritten as

$$\vec{N}(t+1) = \left(\frac{\operatorname{tot}(t+1)}{\operatorname{tot}(t)}\right)\vec{N}(t) = \lambda\vec{N}(t)$$

where

$$\lambda = \frac{\operatorname{tot}(t+1)}{\operatorname{tot}(t)}.$$

But we also have

$$\vec{N}(t+1) = L\vec{N}(t).$$

Comparing our two formulas for $\vec{N}(t)$ gives

$$L\vec{N}(t) = \lambda\vec{N}(t).$$

In this case we let

$$\lambda = \textbf{\textit{growth ratio}}$$
$$r = \lambda - 1 = \textbf{\textit{per capita growth rate.}}$$

This motivates the following. Let L be a square matrix and $\vec{N}$ a vector and λ a number. Then $\vec{N}$ is an ***eigenvector*** of L with ***eigenvalue*** λ if and only if

$$L\vec{N} = \lambda\vec{N}.$$

What this means for us is that $L\vec{N}$ and $\vec{N}$ have the same age distribution. Thus in the context of our class saying that $\vec{N}$ is a eigenvector of L is saying that $\vec{N}$ has the stable age distribution and that the eigenvector λ is the growth ratio.

Let us see what this mean in concrete cases. Let L be the Leslie matrix

$$L = \begin{bmatrix} 0 & 44 & 890 \\ 0.01 & 0 & 0 \\ 0 & 0.8 & 0 \end{bmatrix}$$

and let

$$\vec{N} = \begin{bmatrix} 4 \\ 0.02 \\ 0.009 \end{bmatrix}$$

1. Show that

$$L\vec{N} = \begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix}$$

□

Also note that

$$2\vec{N} = 2\begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix} = \begin{bmatrix} 2(8)8 \\ 2(0.04) \\ 2(0.016) \end{bmatrix} = \begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix}.$$

Thus $\vec{N}$ is a eigenvector for L with eigenvalue 2. Therefore for this Leslie matrix the stable age distributions is

$$\frac{1}{8+.02+.009}\begin{bmatrix} 8 \\ 0.04 \\ 0.016 \end{bmatrix} = \begin{bmatrix} 0.99305 \\ 0.00497 \\ 0.00199 \end{bmatrix},$$

the growth factor is $\lambda = 2$ and $r = \lambda - 1 = 1$ is the per capita growth rate.

2. Let L be the Leslie matrix

$$L = \begin{bmatrix} 0 & 3.9 & 15.25 \\ 0.1 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$$

(a) Show that

$$\vec{N} = \begin{bmatrix} 100 \\ 10 \\ 4 \end{bmatrix}$$

is an eigenvector for L with eigenvalue $\lambda = 1$ and thus that per capita growth rate is $r = \lambda - 1 = 0$.

(b) Find the stable age distribution. *Solution:* It is

$$\begin{bmatrix} 0.87719 \\ 0.08772 \\ 0.03509 \end{bmatrix}$$

(c) If

$$\vec{N}(0) = \begin{bmatrix} 100 \\ 0 \\ 0 \end{bmatrix}$$

find $\vec{N}(1)$, $\vec{N}(2)$, $\vec{N}(5)$ and $\vec{N}(30)$ and the age distribution of these vectors.

Solution:

$$\vec{N}(1) = \begin{bmatrix} 0 \\ 10 \\ 0 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0 \\ 1.0 \\ 0 \end{bmatrix}$$

$$\vec{N}(2) = \begin{bmatrix} 39 \\ 0 \\ 4 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0.90698 \\ 0 \\ 0.09302 \end{bmatrix}$$

$$\vec{N}(5) = \begin{bmatrix} 47.5800 \\ 1.5210 \\ 2.4400 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0.92315 \\ 0.02951 \\ 0.04734 \end{bmatrix}$$

$$\vec{N}(30) = \begin{bmatrix} 38.3362 \\ 3.8266 \\ 1.5343 \end{bmatrix} \quad \text{age distribution is} \quad \begin{bmatrix} 0.87732 \\ 0.08757 \\ 0.03511 \end{bmatrix}$$