Mathematics 172 Homework, March 1, 2019.

We recall form calculus that if h is a small number, then we have the approximation

$$f(a+h) \approx f(a) + f'(a)h.$$

We apply this to giving approximations to solutions to differential equations. If we have a differential equation

$$y'(t) = f(y(t))$$

which we could also write as

$$\frac{dy}{dt} = f(y)$$

and we know the value of $y(t_0)$ for some value t_0 then we can approximate

$$y(t_0 + h) \approx y(t_0) + y'(t_0)h.$$

But since y is the solution to the differential equation $y'(t) = f(y(t))$ we have $y'(t_0) = f(y(t_0))$ and thus

$$y(t_0 + h) \approx y(t_0) + f(y(t_0))h.$$

If we use the notation (really just abbreviations)

$$\begin{aligned} y_0 &= y(t_0) \\ t_1 &= t_0 + h \\ y_1 &= y(t_1) \end{aligned}$$

then we have

$$\begin{aligned} t_1 &= t_0 + h \\ y_1 &\approx y_0 + f(y_0)h. \end{aligned}$$

This is taking a ***step of length*** h ***in Euler's Method***. We can take anther step:

$$\begin{aligned} t_2 &= t_1 + h \\ y_2 &= y_1 + f(y_1)h. \end{aligned}$$

Taking anther step gives

$$\begin{aligned} t_3 &= t_2 + h \\ y_3 &\approx y_2 + f(y_2)h. \end{aligned}$$

In general we will going from the n-th to the $(n+1)$-st step looks like

$$\begin{aligned} t_{n+1} &= t_n + h \\ y_{n+1} &= y_n + f(y_n)h \end{aligned}$$

Then then the approximation

$$y(t_n) \approx y_n$$

should hold.

1. For the initial value problem

$$y' = 1 + .5y \qquad y(0) = 2$$

take two Euler steps of length .25 to estimate $y(.5)$.

Solution: Here we have $h = .25$, $t_0 = 0$ and $y_0 = 2$. The first step is

$$t_1 = t_0 + h = 0 + .25 = .25$$
$$y_1 = 2 + (1 + .5y_0)h = 2 + (1 + .5(2))(.25) = 2.5$$

Then the second step is

$$t_2 = t_1 + h = .5$$
$$y_2 = y_1 + f(y_1)h = 2.5 + (1 + .5(2.5))(.25) = 3.0625$$

2. For the same initial value problem as the last question, estimate $y(.5)$ first by taking 5 steps of length .1 and then 10 steps of length .05.

Solution: I do not really expect any of you to do this, but wanted to show how the solutions converge to the exact solution as the step size gets smaller.

For the five steps of size .1 we get the table:

n	t_n	y_n
0	0.0	2.00000000000000
1	0.1	2.20000000000000
2	0.2	2.41000000000000
3	0.3	2.63050000000000
4	0.4	2.86202500000000
5	0.5	3.10512625000000

This gives the estimate $y(.5) \approx y_5 = 3.10512625000000$

For 10 steps of size .05 the table is

n	t_n	y_n
0	0.000000000000000	2.00000000000000
1	0.050000000000000	2.10000000000000
2	0.100000000000000	2.20250000000000
3	0.150000000000000	2.30756250000000
4	0.200000000000000	2.41525156250000
5	0.250000000000000	2.52563285156250
6	0.300000000000000	2.63877367285156
7	0.350000000000000	2.75474301467285
8	0.400000000000000	2.87361159003967
9	0.450000000000000	2.99545187979066
10	0.500000000000000	3.12033817678543

This gives the estimate $y(.5) \approx y_{10} = 3.12033817678543$

If we really got carried away (and I did) and took 100 steps of size $h = .005$ we would get the estimate $y(.5) \approx y_{100} = 3.13449955495387$

For this equation we can find the exact solution which is

$$y(t) = -2 + 4e^{.5t}$$

and thus the real value is

$$y(.5) = -2 + 4e^{.5(.5)} = 3.13610166675097\ldots$$

So it took about 100 steps to get two decimal places. There are tricks to speed this up, but we will not go into them.