Mathematics 172 Homework, March 26, 2019.

Consider two species, victims and predators. We assume that the only food available to the predators is the victims. And we assume that the victim population is small enough that its growth can be modeled with a constant per capita growth rate. Thus $V = V(t)$ is the number of victims, then without any predators, V satisfies the rate equation

$$\frac{dV}{dt} = rV$$

where r is the per capita growth rate. Likewise we assume that without any victims the predator population size has a constant per capita death rate, that is

$$\frac{dP}{dt} = -dP$$

where d is the per capita death rate. When both victims and predators are present model the interaction between the two as

$$\frac{dV}{dt} = rV - \alpha VP \tag{1}$$

$$\frac{dP}{dt} = -dP + \beta VP \tag{2}$$

where the positive constants α and β depend on how the species interact. (Very very roughly α measures how efficient the predators are, and β measures how nutritious the victims are.)

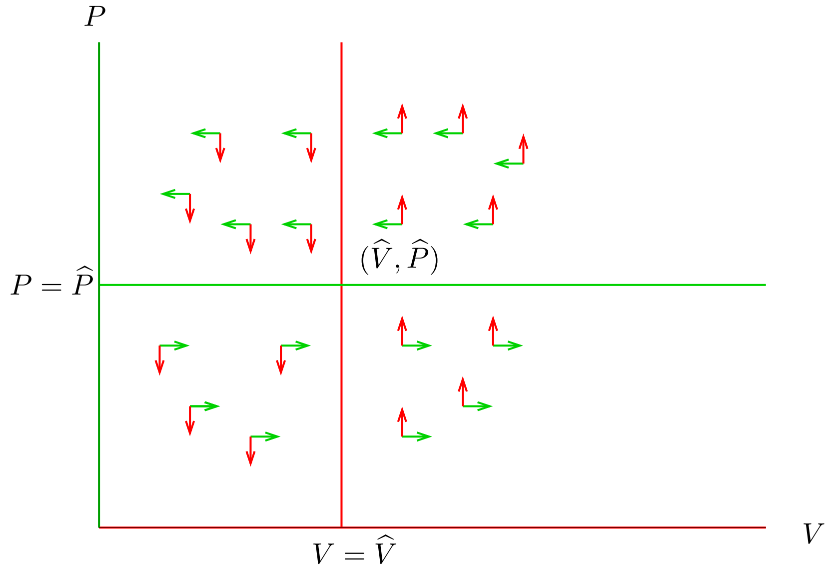
1. Show that the system (3), (4) of equation has two rest points. $(V, P) = (0, 0)$ and $(V, P) = (\widehat{V}, \widehat{P})$ where

$$\widehat{V} = \frac{d}{\beta}$$

$$\widehat{P} = \frac{r}{\alpha}$$

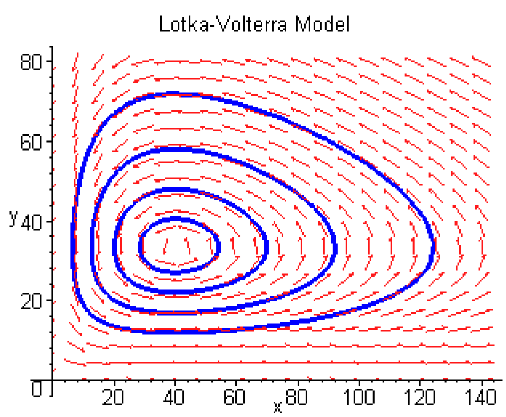
We will call $\widehat{V}$ the ***average number*** of victims and $\widehat{P}$ the ***average number*** of prey.

2. Show that the phase space (that is the V-P plane) for this system looks like Figure 1.



The red is where $dP/dt = 0$ and the green is where $dV/dt = 0$.
Thus there are two rest points: $(0,0)$ and $(\widehat{V}, \widehat{P})$.

Here is a picture that shows the motion of point in more detail. Giving the exact formula for the shape of the ovals is possible, but is requires more calculus than we want to do.



Here is an example of the ovals on which that points $(V(t), P(t))$ move.

3. For the predator-victim system

$$\frac{dV}{dt} = .01V - .002VP$$
$$\frac{dP}{dt} = -.1P + .001VP$$

(a) What is the average number of victims? *Answer:* $\widehat{V} = .1/.001 = 100$

(b) What is the average number of predators? *Answer:* $\widehat{P} = .01/.002 = 5$

(c) If we start with 130 victims and 7 predators, what are $V'(0)$ and $P'(0)$? *Answer:* $V'(0) = -.56$, $P'(0) = .21$

(d) Base on your answer to the last question, is V initially increasing or decreasing. Is P initially increasing or decreasing. *Answer:* V is decreasing and P is increasing.

(e) Using the data from part 3 estimate $V(.2)$ and $P(.2)$. Likewise estimate $V(2)$ and $P(2)$. *Answer:*

$$
\begin{aligned}
V(.2) &\approx V(0) + V'(0).2 = 130 + (-.56)(.2) = 129.888 \\
P(.2) &\approx P(0) + P'(0).2 = 7 + (.21)(.2) = 7.042
\end{aligned}
$$

$$
\begin{aligned}
V(2) &\approx V(0) + V'(0)2 = 130 + (-.56)(2) = 128.88 \\
P(2) &\approx P(0) + P'(0)2 = 7 + (.21)(2) = 7.42
\end{aligned}
$$

4. What happens to the average number of victims if the death rate, $d = .1$ of the prey is doubled to $d = .2$ and the other constants are kept the same? *Answer:* The new $\widehat{V}$ is $\widehat{V} = .2/.001 = 200$, so it is doubled.

What is not so obvious for this system is that the points in the phase space move in closed ovals.

We now change the model a bit. We have been assuming that the with no predators, that the victims grow with a constant per capita growth rate. This is only realistic if the predators keep the victim population well below the carrying capacity so that over crowding does not become a problem. If we assume that the V-population grows logistically when there are no predators, then the equations become

$$
\frac{dV}{dt} = rV\left(1 - \frac{V}{K}\right) - \alpha VP \tag{3}
$$

$$
\frac{dP}{dt} = -dP + \beta VP \tag{4}
$$

To make the algebra and graphing look a little more natural let us change the variables to

$$
\begin{aligned}
x =& V = \text{Number of victums (i.e. prey)} \\
y =& P = \text{Number of predators.}
\end{aligned}
$$

The rate equations then become

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{K}\right) - \alpha xy \tag{5}$$

$$\frac{dy}{dt} = -dy + \beta xy \tag{6}$$

As usual we look for the rest points. First set $\frac{dx}{dt} = 0$. After a little factoring this gives

$$\frac{dx}{dt} = x\left(r\left(1 - \frac{x}{K}\right) - \alpha y\right) = 0.$$

This defines two straight lines. The line $x = 0$, which is just the equation for the y-axis. And also $dx/dt = 0$ on the line

$$r\left(1 - \frac{x}{K}\right) - \alpha y = 0$$

To see what this is, it maybe easier to rearrange it as

$$\frac{rx}{K} + \alpha y = r.$$

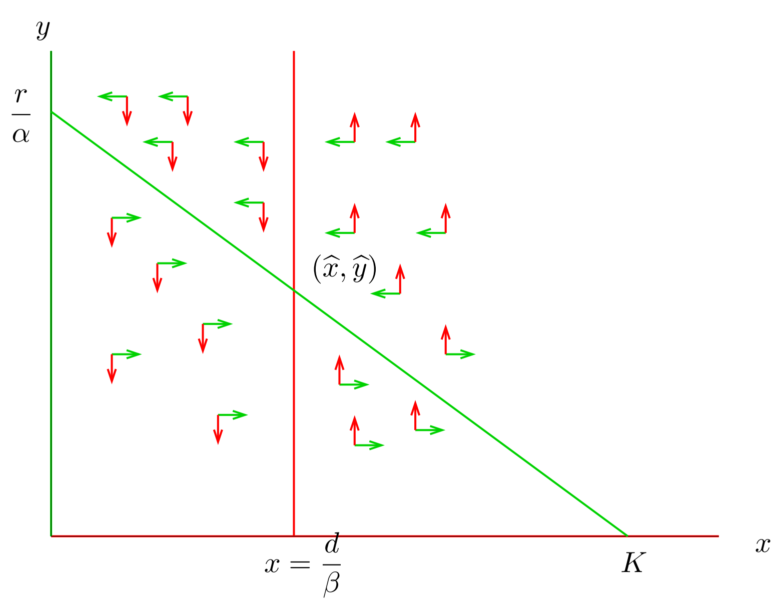
Then letting $y = 0$ and solving for x we see the x-intercept is $(K, 0)$. Setting $x = 0$ and solving for y gives the y-intercept as $(0, r/\alpha)$.

Setting $dy/dt = 0$ gives

$$y(-d + \beta x) = 0$$

which defines the two lines $y = 0$ (the x-axis) and $x = d/\beta$.

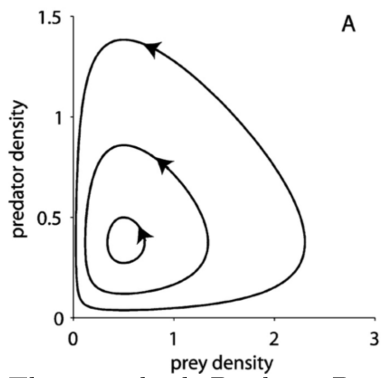
There are two cases. The first is that $d/\beta < K$. Then the phase space looks like:



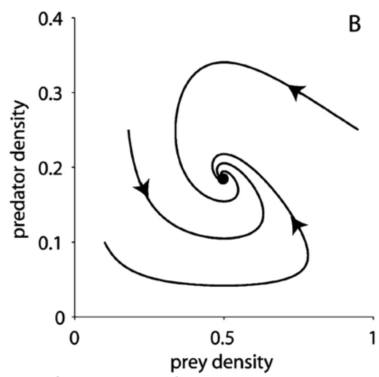
The phase diagram for the equations (5) and (6) in the case where $d/\beta < K$. The points where $dx/dy = 0$ are in green and the points where $dy/dt = 0$ are in red. There are three equilibrium points: $(0,0)$, $(K,0)$ and the point $(\widehat{x}, \widehat{y})$ where

$$\widehat{x} = \frac{d}{\beta}, \qquad \widehat{y} = \frac{r}{\alpha}\left(1 - \frac{d}{K\beta}\right).$$

The points $(\widehat{x}, \widehat{y})$ is stable and the paths that start with positive $x(0)$ and $y(0)$ all spiral into this point.

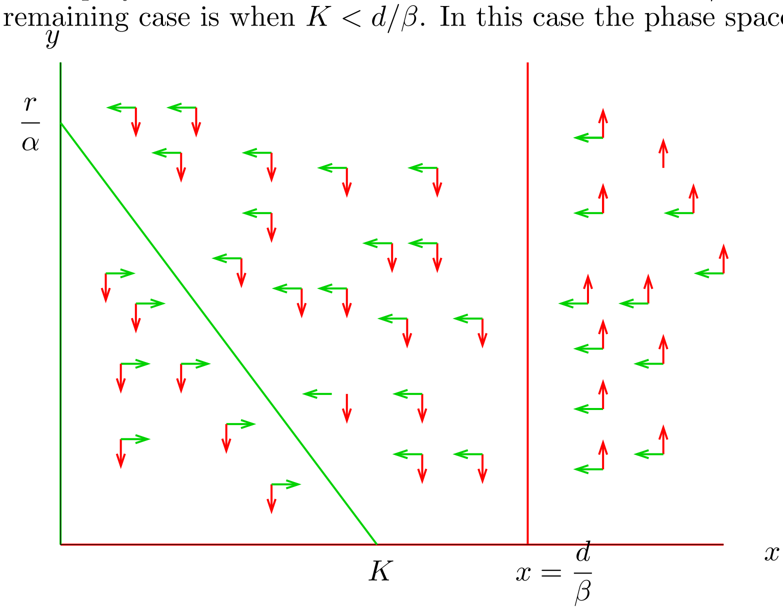


The standard Predator-Prey model with no carrying capacity for the prey.



The Predator-Prey model with logistic growth for the prey in the case that $K > d/\beta$.

The remaining case is when $K < d/\beta$. In this case the phase space looks like:



The phase diagram for the equations (5) and (6) in the case where $d/\beta > K$. The points where $dx/dy = 0$ are in green and the points where $dy/dt = 0$ are in red. There are two equilibrium points: $(0,0)$, $(K,0)$. The point $(K,0)$ is stable. In this case the prey population is not large enough to feed the predators and so the predators die out.

5. For the system

$$\frac{dx}{dt} = .1x\left(1 - \frac{x}{500}\right) - 5xy$$
$$\frac{dy}{dt} = -.2y + .001xy$$

(a) Draw the phase space and discuss what happens in the long run.
(b) If $x(0) = 400$ and $y(0) = 20$ estimate $x(100)$ and $y(100)$.
(c) If $x(0) = 400$ and $y(0) - 0$ estimate $x(100)$ and $y(100)$.