Mathematics 172 Homework, April 10, 2019.

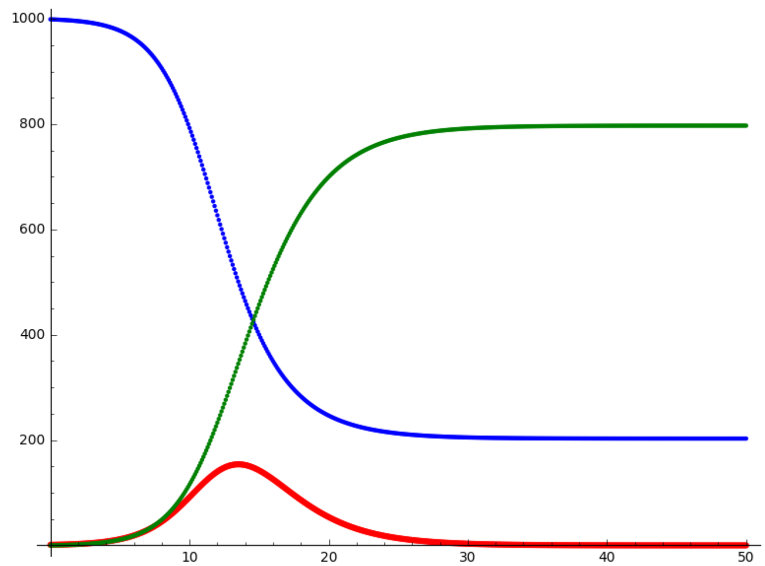
We now look at a variant on the basic SIR model

$$S' = -bSI$$
$$I' = bSI - kI$$
$$R' = kI$$

If we use the values of

$$b = .001$$
$$k = .5$$
$$S(0) = 999$$
$$I(0) = 1$$
$$R(0) = 0$$

we get our standard picture:



One of our assumptions has been that once someone has recovered from the infection that they are immune from reinfection. Let us change this to assume that some proportion, r, of the recovered lose their immunity and become susceptible again. We will model this by putting a couple of extra terms in the equations:

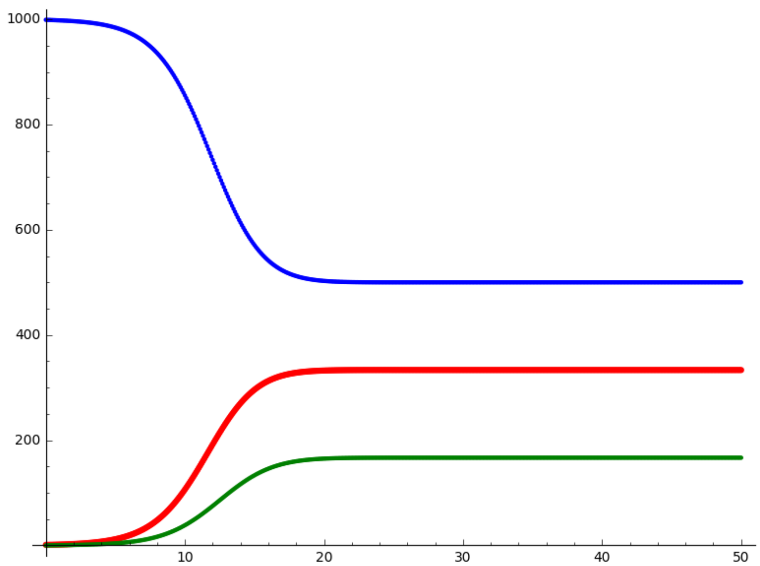
$$S' = -bSI + rR$$
$$I' = bSI - kI$$
$$R' = kI - rR$$

We can think of r as the reinfection rate. Rather like the interpolation of $1/k$ we can think of $1/r$ are being the average length of time it takes a recovered person to lose their immunity and become a susceptible again.

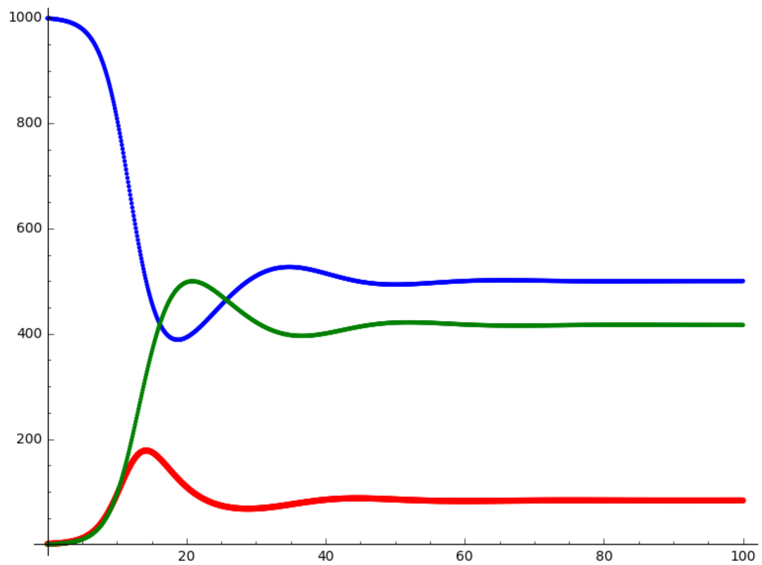
Let us run the equations with the values:

$$
\begin{aligned}
b &= .001\\
k &= .5\\
r &= 1\\
S(0) &= 999\\
I(0) &= 1\\
R(0) &= 1
\end{aligned}
$$

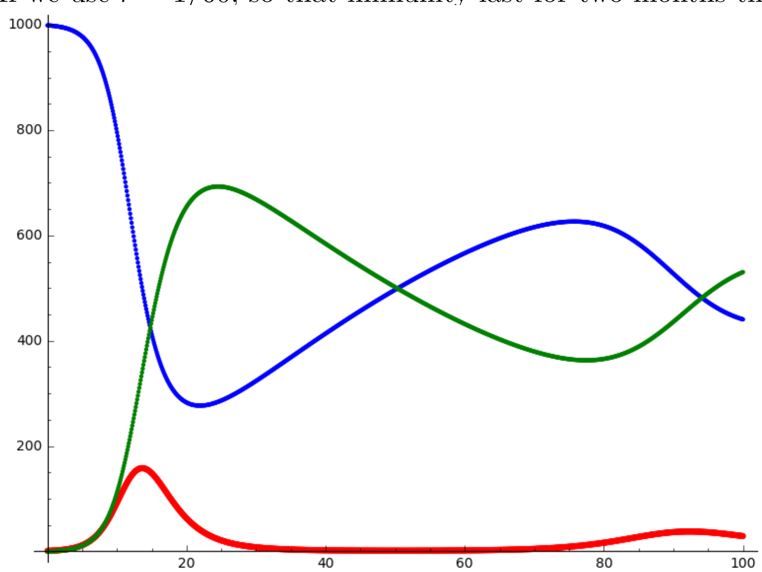
This is saying that a recovereds lose their immunity after just one day. We now have the picture:



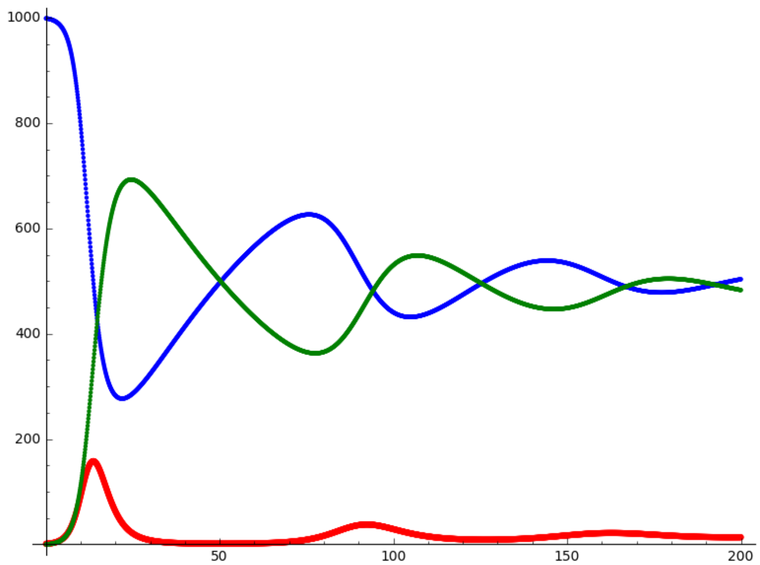
So in this case the infection does not die out, but a in the long run a constant proportion of the population is infected. Keeping everything else the same let us change r to $r = .1$, to that immunity lasts for about 10 days.



So that things osculate a bit before settling down.
If we use $r = 1/60$, so that immunity last for two months the result is



Let us plot this over a longer time period:



All these pictures make it look that for this model the long term behavior is that S, I, and R settle down to fixed values. If this is the case then the values will be rest points of the equations. That is if S_*, I_*, and R_* are the long term values, then they will be solutions to

$$\begin{aligned} S' &= -bSI + rR = 0 \\ I' &= bSI - kI = 0 \\ R' &= kI - rR = 0 \end{aligned}$$

Letting $N = S + I + R$ be the total size of the population and

$$c = \frac{k}{b}$$

be the contact number for the original SIR model some algebra, which I will not inflict on you, gives that the non-zero rest points are (S_*, I_*, R_*) where

$$\begin{aligned} S_* &= c \\ I_* &= \frac{r}{k+r}(N-c) \\ R_* &= \frac{k}{k+r}(N-c) \end{aligned}$$

Thus if the systems stabilizes these are the values of S(big number), I(big number), and R(big number).