Mathematics 172

Quiz 16 **Name:**________________________________

You must show your work to get full credit.

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In this quiz we see how to compute the stable age distribution exactly. Let us use the Leslie matrix from the last quiz, that is

$$L = \begin{bmatrix} 0.0 & 2.4 & 16.0 \\ 0.1 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 \end{bmatrix}$$

Let

$$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

be a vector that has the stable age distribution. Then, to use the terminology we have just introduced, we want to know when $\vec{N}$ is an ***eigenvector*** of L. That is when there is a number number λ (called the ***eigenvalue*** with $L\vec{N} = \lambda\vec{N}$.

1. Show that $L\vec{N} = \lambda\vec{N}$ is the same as This becomes

$$\begin{bmatrix} 2.4n_1 + 16n_2 \\ .1n_1 \\ .5n_2 \end{bmatrix} = \begin{bmatrix} \lambda n_1 \\ \lambda n_2 \\ \lambda n_3 \end{bmatrix}$$

which is equivalent to the three equations

$$2.4n_2 + 16n_3 = \lambda n_1 \tag{1}$$

$$.1n_1 = \lambda n_2 \tag{2}$$

$$.5n_2 = \lambda n_3. \tag{3}$$

2. Use equation (2) to show $n_2 = \dfrac{.1n_1}{\lambda}$. *Solution:* Just divide both sides of (2) by λ.

3. Use the last problem and equation (3) to show $n_3 = \dfrac{.5n_1}{\lambda} = \dfrac{(.5)(.1)n_1}{\lambda^2}$. *Solution:* First divide (3) by λ to get $n_3 = \dfrac{.5n_1}{\lambda}$. Now use the formula for n_2 from the last problem to get

$$n_3 = \frac{.5n_1}{\lambda} = \frac{.5\frac{.1n_1}{\lambda}}{\lambda} = \frac{(.5)(.1)n_1}{\lambda^2}.$$

4. Use the last two problems and equation (1) to show

$$\frac{2.4(.1)}{\lambda^2} + \frac{16(.5)(.1)}{\lambda^3} = 1.$$

Solution: First just plug in the formulas we have for n_2 and n_3 into (1) to get

$$2.4\frac{.1n_1}{\lambda} + 16\frac{(.5)(.1)n_1}{\lambda^2} = \lambda n_1.$$

Each term contains n_1 so we can divide this term out to get

$$2.4\frac{.1}{\lambda} + 16\frac{(.5)(.1)}{\lambda^2} = \lambda.$$

Now dividing by λ gives the desired equation. For use on the next problem note this can be simplified to

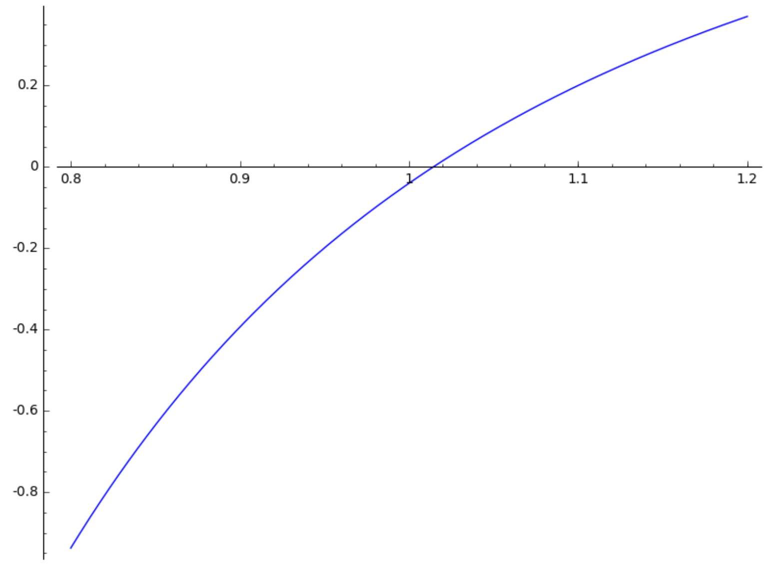
$$\frac{.24}{\lambda^2} + \frac{.8}{\lambda^3} = 1.$$

5. Use your calculator to solve this for λ.

Solution: In your calculator let

\Y$_1$ = 1 -.24/X^2 -.8/X^/3

Use Xmin = .8 and Xmax = 1.2 and then do a ZoomFit to get a graph that looks like



Then do 2nd CALC zero to find that

$$\lambda = 1.0142703$$

6. Now find n_2 and n_3 in terms of n_1. *Solution:* Using our formulas for n_2 and n_3 we find

$$n_2 = \frac{.1n_1}{\lambda} = .0986n_1$$

$$n_3 = \frac{(.1)(.5)n_1}{\lambda^2} = .0486n_1$$

7. Finally give the stable age distribution.

Solution: The total in all the classes is

$$n = n_1 + n_2 + n_3 = n_1 + .0986n_1 + .0486n_1 = 1.1472n_1$$

Thus we have

$$\text{Proportion in Stage 1} = \frac{n_1}{n} = \frac{n_1}{1.1472n_1} = .872 = 87.2\%$$

$$\text{Proportion in Stage 2} = \frac{n_1}{n} = \frac{.0986n_1}{1.1472n_1} = .086 = 8.6\%$$

$$\text{Proportion in Stage 3} = \frac{n_1}{n} = \frac{.0486n_1}{1.1472n_1} = .042 = 4.2\%$$

8. Finally, what are the growth ratio and the per capita growth rate?.

Solution: The growth ratio is the number λ we found above: $\lambda = 1.0142703$. The per capita is $r = \lambda - 1 = .0142703$.

Now let us do this all over again for a general Leslie matrix,

$$L = \begin{bmatrix} f_1 & f_2 & f_3 \\ p_1 & 0 & 0 \\ 0 & p_2 0 & \end{bmatrix}$$

Let

$$\vec{N} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

be an eigenvector with eigenvalue λ. Then

$$L\vec{N} = \lambda\vec{N}.$$

9. Show this leads to the equations

$$f_1 n_1 + n_2 f_2 + n_3 f_3 = \lambda n_1 \tag{4}$$

$$p_1 n_1 = \lambda n_2 \tag{5}$$

$$p_2 n_2 = \lambda n_3 \tag{6}$$

Solution: This is either done by matrix multiplication or by using a loop diagram as we have done in class many times.

10. Do calculations as above to show

$$n_2 = \frac{p_1 n_1}{\lambda} \quad \text{and} \quad n_3 = \frac{p_1 p_2 f_3}{\lambda^2}.$$

Solution: Dividing equation (5) by λ gives

$$n_2 = \frac{p_1 n_1}{\lambda}$$

which is one of the two equation we want. To get the section divide equation (6) by λ to get

$$n_3 = \frac{p_2}{\lambda} n_2.$$

Now use the formula for n_2 we have just found to get

$$n_3 = \frac{p_2}{\lambda}\left(\frac{p_1 n_1}{\lambda}\right) = \frac{p_1 p_2 n_1}{\lambda^2}$$

as required.

11. Use these formulas for n_1 and n_2 in equation (4) to get

$$f_1 + \frac{p_1 f_2}{\lambda^2} + \frac{p_1 p_2 f_3}{\lambda^3} = \lambda.$$

Solution: Do the substations for n_2 and n_3 in (4) gives

$$f_1 n_1 + \left(\frac{p_1 n_1}{\lambda}\right) f_2 + \left(\frac{p_1 p_2 n_1}{\lambda^2}\right) f_3 = \lambda n_1$$

Dividing by the common factor gets rid of the n_1 factors to give

$$f_1 + \left(\frac{p_1}{\lambda}\right) f_2 + \left(\frac{p_1 p_2}{\lambda^2}\right) f_3 = \lambda$$

which simplifies down to the equation we want.

12. Divide the last equation by λ to get

$$\frac{f_1}{\lambda} + \frac{p_1 f_2}{\lambda^2} + \frac{p_1 p_2 f_3}{\lambda^3} = 1.$$

This is the ***Euler-Lotka equation***. It has a unique positive solution, which is the growth of the stable age distribution of the population.

Solution: Just divide by *lambda*.

13. Now that we have λ and also the formulas for n_2 and n_3, use this to find the stable age distribution.

Solution: We have that the total (the sum of the entries in $\vec{N}$) is

$$\begin{aligned} n &= n_1 + n_2 + n_3 \\ &= n_1 + \frac{p_1 n_1}{\lambda} + \frac{p_1 p_2 n_1}{\lambda^2} \\ &= n_1 \left(1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}\right) \end{aligned}$$

Therefore we have

$$\text{Proportion in Stage 1} = \frac{n_1}{n} = \frac{n_1}{n_1\left(1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}\right)} = \frac{1}{1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}}$$

$$\text{Proportion in Stage 2} = \frac{n_2}{n} = \frac{\frac{p_1 n_1}{\lambda}}{n_1\left(1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}\right)} = \frac{\frac{p_1}{\lambda}}{1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}}$$

$$\text{Proportion in Stage 3} = \frac{n_3}{n} = \frac{\frac{p_1 n_1}{\lambda}}{n_1\left(1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}\right)} = \frac{\frac{p_1 p_2}{\lambda^2}}{1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}}$$

This may be easier to take in when written in vector notation:

$$\text{Vector of the stable age distribution} = \frac{1}{1 + \frac{p_1}{\lambda} + \frac{p_1 p_2}{\lambda^2}} \begin{bmatrix} 1 \\ \frac{p_1}{\lambda} \\ \frac{p_1 p_2}{\lambda^2} \end{bmatrix}$$