Mathematics 555 Test 2 Name: Answer Key

1. (a) Define what it means for φ to be a step function on the interval $[a,b]$.

Solution: There is a partition $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$ of $[a,b]$ such that φ is constant on each of the open intervals (x_{j-1}, x_j) for $j = 1, 2, \dots, n$. $\square$

(b) Assume we we have defined the integral, $\int_a^b \varphi(x)\,dx$, for step functions, φ. Let $f\colon [a,b] \to \mathbf{R}$ be a bounded function. Define the upper and lower integrals of f.

Solution:

$$\overline{\int_a^b} f(x)\,dx = \inf\left\{ \int_a^b \psi(x)\,dx : \psi \text{ is a step function and } f \le \psi \right\}$$

$$\underline{\int_a^b} f(x)\,dx = \sup\left\{ \int_a^b \varphi(x)\,dx : \varphi \text{ is a step function and } \varphi \le f \right\} \quad \square$$

(c) Give an example of a function $f\colon [a,b] \to \mathbf{R}$ where

$$\underline{\int_a^b} f(x)\,dx = 0,$$

$$\overline{\int_a^b} f(x)\,dx = b - a.$$

You do not have to prove your example works.

Solution: One of our standard perverse functions does the trick:

$$f(x) = \begin{cases} 1, & x \text{ is a rational number;} \\ 0, & x \text{ is an irrational number.} \end{cases}$$

2. Let $|x| < 1/2$. Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} \frac{5x^{2k+1}}{4^k}$.

Solution: We have

$$\sum_{k=0}^{\infty} \frac{5x^{2k+1}}{4^k} = 5x + \frac{5x^3}{4} + \frac{5x^5}{4^3} + \frac{5x^7}{4^3} + \cdots$$

This is a geometric series with first term $5x$ and ratio $r = x^2/4$. As $|x| < 1/2$ the ratio is less than 1 (in fact we would be ok if $|x| < 2$)

and thus the series converges. Therefore

$$\sum_{k=0}^{\infty} \frac{5x^{2k+1}}{4^k} = \frac{\text{first}}{1 - \text{ratio}} = \frac{5x}{1 - x^2/4} = \frac{20x}{4 - x^2}.$$

□

3. (a) Give the definition of the natural logarithm, $\ln(x)$, in terms of an integral.

Solution: The definition is

$$\ln(x) = \int_1^x \frac{dt}{t}.$$

(The function $1/t$ is continuous on $(0, \infty)$ thus the integral exists.) □

(b) Use this definition the change of variable formula for integrals to show that for $a, b > 0$

$$\ln(ab) = \ln(a) + \ln(b).$$

Solution:

$$\begin{aligned}
\ln(ab) &= \int_1^{ab} \frac{dx}{x} \\
&= \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x} && \left(\begin{array}{l}\text{Let } x = au, \text{ then } dx = a\,du. \\ \text{Then } x = a \text{ implies } u = 1 \\ \text{and } x = ab \text{ implies } u = b.\end{array}\right) \\
&= \int_1^a \frac{dx}{x} + \int_1^b \frac{a\,du}{au} \\
&= \int_1^a \frac{dx}{x} + \int_1^b \frac{du}{u} \\
&= \ln(a) + \ln(b).
\end{aligned}$$

□

4. (a) Recall one version of the Fundamental Theorem of Calculus is that if $f\colon [a,b] \to \mathbf{R}$ is continuous, F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

and $x_0 \in (a,b)$, then

$$F'(x_0) = f(x_0).$$

Prove this.

Solution: Let $\varepsilon > 0$. Then, as f is continuous at x_0, there is a $\delta > 0$ such that

$$|x - x_0| < \delta \quad \text{implies} \quad |f(x) - f(x_0)| < \varepsilon.$$

Then $0 < |x - x_0| < \delta$ implies

$$\begin{aligned}
\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| &= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - f(x_0) \cdot 1 \right| \\
&= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - f(x_0) \left(\frac{1}{x - x_0} \int_{x_0}^x 1\,dt \right) \right| \\
&= \left| \frac{1}{x - x_0} \int_{x_0}^x f(t)\,dt - \frac{1}{x - x_0} \int_{x_0}^x f(x_0)\,dt \right| \\
&= \left| \frac{1}{x - x_0} \int_{x_0}^x (f(t) - f(x_0))\,dt \right| \\
&\le \frac{1}{x - x_0} \int_{x_0}^x |f(t) - f(x_0)|\,dt \\
&< \frac{1}{x - x_0} \int_{x_0}^x \varepsilon\,dt \\
&= \varepsilon.
\end{aligned}$$

Here we have used in the integral $\int_{x_0}^x |f(t) - f(x_0)|\,dt$ that t is between x and x_0 thus $|t - x_0| \le |x - x_0| < \delta$, and $|t - x_0| < \delta$ implies $|f(t) - f(x_0)| < \varepsilon$. To summarize this calculation we have

$$0 < |x - x_0| < \delta \quad \text{implies} \quad \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \varepsilon.$$

That is $F'(x_0)$ exists and is given by

$$F'(x_0) = \lim_{x \to x_0} \frac{F(x) - F(x_0)}{x - x_0} = f(x_0).$$

□

(b) Use part (a) to prove that if $f\colon [a,b] \to \mathbf{R}$ is continuous and there is a function F with $F'(x) = f(x)$ on $[a,b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Solution: Let

$$G(x) = F(x) - \int_a^x f(t)\,dt.$$

Then

$$G'(x) = F'(x) - \frac{d}{dx}\int_a^x f(t)\,dt = f(x) - f(x) = 0.$$

As the derivative of G is zero on (a,b) and G is continuous this implies G is a constant. Say $G(x) = c$. Then

$$F(x) - \int_a^x f(t)\,dt = c.$$

Letting $x = a$ and using $\int_a^a f(t)\,dt = 0$ gives

$$F(a) = -c$$

and so

$$F(x) - \int_a^x f(t)\,dt = -F(a),$$

which can be rearranged as

$$\int_a^x f(t)\,dt = F(x) - F(a).$$

Letting $x = b$ completes the proof. □

5. Which of the following series converge. Justify your answer.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+n}{n^5-9}$.

Solution: Note

$$\lim_{n\to\infty} \frac{\dfrac{n^3+n}{n^5-9}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^2(n^3+n)}{n^5-9} = 1.$$

Therefore the series ***converges*** by limit comparison to the convergent series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (A little more should be said as the limit comparison theorem compares series of positive terms. However the given series does have a negative term when $n = 1$. But we can just do the comparison with the seres $\sum_{n=2}^{\infty} \frac{n^3+n}{n^5-9}$ starting at $k = 2$.) □

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1+\sqrt{n}}{200n+1}$.

Solution: This time we have

$$\lim_{n\to\infty} \frac{\dfrac{1+\sqrt{n}}{200n+1}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}(1+\sqrt{n})}{200n+1} = \frac{1}{200}.$$

Therefore this series ***diverges*** by limit comparison to the divergent series $\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$. □

(c) $\displaystyle\sum_{k=1}^{\infty} \sin(2^{-k})$.

Solution: If $0 < x < \pi/2$ we have have by the Mean Value Theorem there is a ξ between 0 and x with

$$0 < \sin(x) = \sin(x) - \sin(0) = \cos(\xi)(x-0) = \cos(\xi)x < x.$$

Therefore for $k = 1, 2, 3, \ldots$

$$0 < \sin(2^{-k}) < 2^{-k}.$$

The series $\sum_{k=1}^{\infty} 2^{-k}$ is a convergent geometric series and therefore $\sum_{k=1}^{\infty} \sin(2^{-k})$ ***converges*** by comparison to this geometric series. □