Mathematics 555 Test 3 Name: Answer Key

1. (a) Define what it mean for the sequence of function $\langle K_n \rangle_{n=1}^{\infty}$ to be a ***Dirac Sequence.***

Solution. Each of the functions K_n is Riemannian integratable on $(-\infty, \infty)$ and satisfies the following three properties:

(i) $K_n(x) \geq 0$ for all n and x.

(ii) $\displaystyle\int_{-\infty}^{\infty} K_n(x)\,dx = 1$ for all n.

(iii) For all $\delta > 0$

$$\lim_{n\to\infty} \int_{|y|\geq\delta} K_n(x)\,dx = 0.$$

□

(b) Let $f\colon \mathbf{R} \to \mathbf{R}$ be a bounded uniformly continuous function and let

$$f_n(x) = \int_{-\infty}^{\infty} f(x-y)K_n(y)\,dy$$

prove that $\lim\limits_{n\to\infty} f_n(x) = f(x)$ uniformly.

Solution. See class notes. (Everyone got this one correct.) □

2. (a) State the Weierstrass M-test.

Solution. Let $f_1, f_2, f_3, \ldots X \to \mathbf{R}$ be a sequence of functions from a set X to the real numbers. Assume there are constants M_k such that $|f_k(x)| \leq M_k$ for all x and k and that

$$\sum_{k=1} M_k < \infty.$$

Then the series

$$\sum_{k=1}^{\infty} f_k(x)$$

converges both absolutely and uniformly. □

(b) Define $f\colon \mathbf{R} \to \mathbf{R}$ by the series

$$f(x) = \sum_{k=1}^{\infty} \frac{2^k}{4^k + x^4}.$$

Prove $f(x) \leq 1$ and that f is uniformly continuous.

Solution. Let $f_k\colon \mathbf{R} \to \mathbf{R}$ be the function

$$f_k(x) = \frac{2^k}{4^k + x^4}.$$

Then

$$0 \le f_k(x) = \frac{2^k}{4^k + x^4} \le \frac{2^k}{4^k + 0} = \frac{1}{2^k}.$$

Thus if $M_k = \dfrac{1}{2^k}$ we have $|f_k(x)| \le M_k$ and $\sum_{k=1}^\infty M_k$ is a convergent geometric series. Therefore the series $\sum_{k=1}^\infty f_k(x)$ is absolutely and uniformly convergent. Therefore the sum, $f(x)$, is the uniform limit of continuous functions and thus f is continuous.

The only thing left is to show f is uniformly continuous. To do this it is enough to show that each f_k is uniformly continuous, for then the partial sums for f will be uniformly continuous and therefore f will be a uniformly convergent limit of uniformly continuous functions and therefore will itself be uniformly continuous. By the Mean Value Theorem it is enough to show the derivative $|f_k'(x)|$ is bounded, for then f_k will be Lipschitz and therefore uniformly continuous.

$$f_k'(x) = \frac{-4(2^k)x^3}{(4^k + x^4)^2}.$$

Then if $|x| \le 1$ we have

$$|f_k'(x)| = \frac{4(2^k)|x|^3}{(4^k + x^4)^2} \le \frac{4(2^k)}{(4^k + 0^4)^2} = \frac{4}{8^k}.$$

If $|x| \ge 1$

$$|f_k'(x)| = \frac{4(2^k)|x|^3}{(4^k + x^4)(4^k + x^k)} \le \frac{4(2^k)|x|^3}{(4^k + 0^4)(0 + x^4)} = \frac{4(2^k)}{4^k|x|} \le \frac{4}{2^k}.$$

Putting these inequalities together gives

$$|f_k'(x)| \le \max\left\{\frac{4}{8^k}, \frac{4}{2^k}\right\} = \frac{4}{2^k}$$

which shows $|f_k'|$ is bounded and finishes the proof. □

3. (a) Give our official definition for $\sin(x)$.

Solution.

$$\sin(x) = \sum_{k=0}^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$

□

(b) Find the power series for $\displaystyle\int_0^x \sin(t^3)\,dt$ about the point $x = 0$.

Solution. We do this by letting $x = t^3$ in the series for $\sin(x)$ and using that we can integrate power series term by term:

$$\begin{aligned}\int_0^x \sin(t^3)\,dt &= \int_0^x \left(\sum_{k=0}^{\infty}(-1)^k \frac{t^{3(2k+1)}}{(2k+1)!}\right) dt \\ &= \sum_{k=0}^{\infty}(-1)^k \int_0^x \frac{t^{3(2k+1)}}{(2k+1)!}\,dt \\ &= \sum_{k=0}^{\infty}(-1)^k \frac{x^{6k+4}}{(2k+1)!(6k+4)}.\end{aligned}$$

□

4. Find the power series expansion of $f(x) = \dfrac{1}{x}$ about the point $x = 2$ and give its radius of convergence.

Solution 1. We know that the series is

$$f(x) = \sum_{k=0}^{\infty} a_k (x-2)^k$$

where

$$a_k = \frac{f^{(k)}(2)}{k!}.$$

Taking the first several derivatives gives

$$\begin{aligned}f(x) &= x^{-1} \\ f'(x) &= -x^{-2} \\ f''(x) &= (-1)(-2)x^{-3} \\ f'''(x) &= (-1)(-2)(-3)x^{-4} \\ f^{(4)}(x) &= (-1)(-2)(-3)(-4)x^{-5} \\ f^{(5)}(x) &= (-1)(-2)(-3)(-4)(-5)x^{-6}\end{aligned}$$

at which point we see the patten:

$$f^{(k)}(x) = (-1)^k k!\, x^{-(k+1)}.$$

Therefore

$$a_k = \frac{f^{(k)}(2)}{k!} = \frac{(-1)^k k! 2^{-(k+1)}}{k!} = \frac{(-1)^k}{2^{k+1}}$$

and the required series is

$$f(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{2^{k+1}}.$$

Using the ratio test we easily see that the radius of convergence is $R = 2$. ☐

Solution 2. We are experts on the geometric series

$$\frac{1}{1+y} = \sum_{k=0}^{\infty} (-1)^k y^k.$$

So we try to reduce the case of the expansion of $f(x) = 1/x$ around $x = 2$ to this series.

$$\begin{aligned} \frac{1}{x} &= \frac{1}{2 + (x-2)} \\ &= \frac{1}{2}\left(\frac{1}{1 + \left(\frac{x-2}{2}\right)}\right) \\ &= \frac{1}{2}\sum_{k=0}^{\infty} (-1)^k \left(\frac{x-2}{2}\right)^k \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{2^k}. \end{aligned}$$

And again the ratio test (or root test) can be used to show that the radius of convergence is $R = 2$. ☐

5. Give examples a power series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

such that

(a) The radius of convergence is $R = 1$ and $f(x)$ converges at both the points 1 and -1.

Solution. About the easiest example is

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{(k+1)^2}.$$

☐

(b) The radius of convergence is $R = 1$ and $f(x)$ diverges at both the points 1 and -1.

Solution. Our basic geometric series

$$f(x) = \sum_{k=0}^{\infty} x^k$$

works here. □

(c) The radius of convergence is $R = 2$ and $f(x)$ diverges at 2, but converges at -2.

Solution. Let

$$f(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^k}{2^k(k+1)}.$$

The ratio test tells us the radius of convergence is $R = 2$. When $x = 2$ the series is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1}$$

which is a convergent alternating series.

When $x = -2$ the series is

$$\sum_{k=0}^{\infty} \frac{1}{k+1}$$

which diverges. □