Solutions to take home portion of final.

(Solutions to in class portion start on page 5 below.)

We start by extending our definition of $\sup(S)$ and $\inf(S)$ for nonempty subsets of $\mathbf{R}$. If S is bounded above, the $\sup(S)$ is the usual ***least upper bound*** or ***supremum***. If S is not bounded above set $\sup(S) = +\infty$. Likewise if S is bounded below, then $\inf(S)$ is the usual ***greatest lower bound*** or ***infinmum***, if S is not bounded below, then $\inf(S) = -\infty$.

We also want to extend the definition of the limit of a sequence.

Definition 1. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\lim_{k \to \infty} a_k = +\infty$$

if and only if for all real numbers B there is a N such that

$$k \geq N \quad \text{implies} \quad a_k > B.$$

$\square$

Definition 2. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers. Then

$$\lim_{k \to \infty} a_k = -\infty$$

if and only if for all real numbers A there is a N such that

$$k \geq N \quad \text{implies} \quad a_k < A.$$

$\square$

Definition 3. The ***extended real numbers*** is the set $\mathbf{R} \cup \{-\infty, +\infty\}$. That is the extended real numbers is just the usual real numbers with the two values $-\infty$ and $+\infty$ thrown in. $\square$

Note that now every nonempty subset of $\mathbf{R}$ has an sup and inf in the extended real numbers.

One of the more useful results from last term was that any bounded monotone sequence is convergent. With the above definition we can drop the requirement that the sequence be bounded.

Proposition 4. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a monotone sequence. Then it has a limit in the extended real numbers.*

Proof. Assume that $\langle a_k \rangle_{k=1}^{\infty}$ is monotone increasing. If the sequence is bounded above, then we saw last term that it it converged to $\sup\{a_1, a_2, a_3, \ldots\}$. So assume that it is not bounded above. Then for any real number B, there is some N with $a_N > B$. But $\langle a_k \rangle_{k=1}^{\infty}$ is monotone increasing and therefore if $k \geq N$ we have $a_k \geq a_N > B$, and therefore $\lim_{k \to \infty} a_k = +\infty$. $\square$

Lemma 5. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of read numbers and set*

$$S_n = \sup\left(\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right).$$

Then $\langle S_n \rangle_{n=1}^{\infty}$ is a monotone decreasing sequence.

Proof. The number S_n is upper bound for the set

$$\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

and

$$\{a_{n+1}, a_{n+1}, a_{n+3}, \ldots\} \subseteq \{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$

Therefore S_n is also an upper bound for

$$\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\}$$

But $S_{n+1} = \sup(\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\})$ is the least upper bound for this set and therefore $S_{n+1} \leq S_n$. □

Likewise we have

Lemma 6. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of read numbers and set*

$$I_n = \sup\left(\{a_n, a_{n+1}, a_{n+2}, \ldots\}\right).$$

Then $\langle I_n \rangle_{n=1}^{\infty}$ is a monotone increasing sequence.

Problem 1. Prove this. □

Solution. If A and B are subsets of $\mathbf{R}$ with $A \subseteq B$, then $\sup(A) \leq \sup(B)$ (as any upper bound for B is an upper bound for A). As $\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\} \subseteq \{a_n, a_{n+1}, a_{n+2}\}$ the lemma follows. □

The last two lemmas and Proposition 4 imply that the following two limits exist in the extended real numbers.

$$\lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$
$$\lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

Therefore the following definition makes sense.

Definition 7. Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers.

$$\limsup_{k\to\infty} a_k = \lim_{n\to\infty} \sup\{a_n, a_{n+1}, a_{n+1}, \ldots\}$$
$$\liminf_{k\to\infty} a_k = \lim_{n\to\infty} \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}.$$ □

Proposition 8. *For any sequence $\langle a_k \rangle_{k=1}^{\infty}$ show that*

$$\liminf_{k\to\infty} a_k \leq \limsup_{k\to\infty} a_k.$$

Problem 2. Prove this. □

Solution. For each n we have

$$\inf\{a_n, a_{n+1}, a_{n+2}, \ldots\} \leq \sup\{a_n, a_{n+1}, a_{n+1}, \ldots\}.$$

The result follows by taking the limit at $n \to \infty$. □

Problem 3. Find the following:

(a) $\liminf_{n\to\infty}(3 + (-1)^n)$.

(b) $\liminf_{n\to\infty}(-2)^n$.

Solution. (a) If $a_n = 3+(-1)^n$ then $a_n = 4$ for even n and $a_n = 2$ for odd n. The for all n the set $\{a_n, a_{n+1}, a_{n+2}, \ldots\}$ is just the two element set $\{2, 4\}$. Therefore $\liminf_{n\to\infty} a_n = 2$.

(b) Let b be any real number and n a positive integer. Then for each n there is an odd $m \geq n$ such $(-2)^m < b$. This shows that $\{(-2)^n, (-2)^{n+1}, (-2)^{n+2}, \ldots\}$ has no lower bound. Therefore $\liminf_{n\to\infty}(-2)^n = -\infty$. $\square$

Problem 4. Give examples of sequences $\langle a_k\rangle_{k=1}^{\infty}$ such that

(a) $\liminf_{k\to\infty} a_k = -1$ and $\limsup_{k\to\infty} a_k = +1$.

(b) $\liminf_{k\to\infty} a_k = 0$ and $\limsup_{k\to\infty} a_k = +\infty$.

Solution. (a) The sequence $a_n = (-1)^n$ does the trick.

(b) Let $a_n = (1 + (-1)^n)n$. Then

$$a_n = \begin{cases} 2n, & n \text{ is even;} \\ 0, & n \text{ is odd.} \end{cases}$$

which has the required lim inf and lim sup. $\square$

Proposition 9. *If $\langle a_k\rangle_{k=1}^{\infty}$ is a sequence with such that*

$$\lim_{k\to\infty} a_k = L$$

exists, then

$$\limsup_{n\to\infty} a_k = \liminf_{k\to\infty} a_k = L.$$

Problem 5. Prove this. $\square$

Solution. We first consider the case of $L \neq \pm\infty$. Let $\varepsilon > 0$. Then $\lim_{n\to\infty} a_n = L$ implies there is a N such that

$$n \geq N \quad \text{implies} \quad |a_n - L| < \varepsilon.$$

Therefore $a_n > L - \varepsilon$ for $n \geq N$ which yields that

$$n \geq N \quad \text{implies} \quad L - \varepsilon \leq \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

which implies

$$L \leq \liminf_{\to\infty} a_n.$$

Likewise $n \geq N$ implies $a_n \leq L + \varepsilon$ which gives that

$$L \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq L$$

which proves $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = L$

If $L = \infty$, then $\lim_{\to\infty} a_n = \infty$ which by definition implies that for all $B \in \mathbf{R}$ there is a N such that

$$n \geq \quad \text{implies} \quad a_n \geq B.$$

This in turn implies for $n \geq N$

$$B \leq \inf\{a_n, a_{n+1}, a_{n+2}\} \leq \sup\{a_n, a_{n+1}, a_{n+2}\}$$

which shows that $\liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n = \infty$.

The case of $L = -\infty$ is similar. $\square$

Proposition 10. *If*

$$\liminf_{k\to\infty} a_k = \limsup_{k\to\infty} a_k$$

and this number is finite, then

$$\lim_{k\to\infty} a_k$$

exists.

Problem 6. Prove this. $\square$

Solution. Let $L = \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$ be the common value of the lim inf and lim sup. First assume L is finite (that is $L \neq \pm\infty$). Let $\varepsilon > 0$. Then $\liminf_{n\to\infty} a_n = L$ implies there is a N_1 such that

$$n \geq N_1 \quad \text{implies} \quad L - \varepsilon < a_n.$$

Likewise $\limsup_{n\to\infty} = L$ implies there is a N_2 such that

$$n \geq N_2 \quad \text{implies} \quad a_n < L + \varepsilon.$$

Therefore if $N = \max\{N_1, N_2\}$ we have

$$n \geq N \quad \text{implies} \quad L - \varepsilon < a_n < L + \varepsilon.$$

That is $n \geq N$ implies $|a_n - L| < \varepsilon$ and thus $\lim_{n\to\infty} a_n = L$.

Now assume $L = \infty$. Then $\liminf_{n\to\infty} a_n = \infty$. Then $\lim_{n\to\infty} a_n = \infty$. This means for all $B \in \mathbf{R}$ there is a n such that

$$n \geq N \quad \text{implies} \quad a_n \geq B.$$

This implies that for $n \geq N$ the inequality

$$B \leq \inf\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$

which verifies the definition of $\lim_{n\to\infty} a_n = \infty = L$.

A similar argument covers the case of $L = -\infty$. $\square$

Proposition 11. *If $\langle a_k \rangle_{k=1}^{\infty}$ and $\langle b_k \rangle_{k=1}^{\infty}$ are two sequences of real numbers, then*

$$\limsup_{k\to\infty}(a_k + b_k) \leq \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k.$$

Problem 7. Prove this and give an example where

$$\limsup_{k\to\infty}(a_k + b_k) < \limsup_{k\to\infty} a_k + \limsup_{k\to\infty} b_k. \quad \square$$

Solution. For each n let

$$A_n = \sup\{a_n, a_{n+1}, a_{n+2}, \ldots\}$$
$$B_n = \sup\{b_n, b_{n+1}, b_{n+2}, \ldots\}.$$

Then if $m \geq n$ we have $a_m \leq A_n$ and $b_m \leq B_n$ and thus

$$a_m + b_m \leq A_n + B_n.$$

Thus

$$\sup\{a_n + b_n, a_{n+1} + b_{n+1}, a_{n+2} + b_{n+2}, \ldots\} \le A_n + B_n$$

and therefore

$$\begin{aligned}
\limsup_{n\to\infty}(a_n + b_n) &= \lim_{n\to\infty} \left(\sup\{a_n + b_n, a_{n+1} + b_{n+1}, a_{n+2} + b_{n+2}, \ldots\}\right) \\
&\le \lim_{n\to\infty} (A_n + B_n) \\
&= \limsup_{n\to\infty} b_n + \limsup_{n\to\infty} a_n
\end{aligned}$$

For the example let $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $a_n + b_n = 0$ and so

$$\limsup_{n\to\infty}(a_n + b_n) = 0 < 2 = \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

□

Solutions to in class part of final.

1. (a) State the ***Mean Value Theorem***.

Solution. Let $f\colon [a,b] \to \mathbf{R}$ be a function that is continuous on $[a,b]$ and differentiable on (a,b). Then there is a $\xi \in (a,b)$ such that the equality

$$f(b) - f(a) = f'(\xi)(b-a)$$

holds. □

(b) Let $h\colon \mathbf{R} \to \mathbf{R}$ be a function that satisfies

$$h'(x) = \frac{3h(x)^2}{1 + h(x)^2}.$$

Prove for all $a, b \in \mathbf{R}$ that

$$|h(b) - h(a)| \le 3|b-a|.$$

Solution. Note that

$$0 \le \frac{3h(x)^2}{1 + h(x)^2} \le \frac{3h(x)^2}{0 + h(x)^3} = 3.$$

Thus by the equation satisfied by h we have $|h'(x)| \le 3$ for all x. Therefore by the Mean Value Theorem there is a ξ such that

$$|h(b) - h(a)| = |h'(\xi)||b-a| \le 3|b-a|.$$

□

2. (a) State Taylor's Theorem with Lagrange's form of the remainder.

Solution. Let f be $n+1$ times differentiable on the open interval (α, β). Then for $a, x \in (\alpha, \beta)$ there is ξ between a and x such that the equality

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

holds. □

(b) Let $f\colon \mathbf{R}\to\mathbf{R}$ be a twice differentiable function and assume $f''(x)<0$ for all $x\in\mathbf{R}$. Show for all $a,x\in\mathbf{R}$

$$f(x)\le f(a)+f'(a)(x-a).$$

(That is the graph of f is below all its tangent lines.)

Solution. We use Taylor's Theorem with $n=1$. Then for all $a,x\in\mathbf{R}$ there is a ξ between a and x with

$$\begin{aligned} f(x) &= f(a)+f'(a)(x-a)+\frac{f''(\xi)}{2}(x-a)^2 \\ &\le f(a)+f'(a)(x-a) \end{aligned}$$

as $f''(\xi)\le 0$. □

3. Let $f_1,f_2,f_3,\ldots X\to Y$ be a map between metric spaces and $M>0$ a constant such that for all $x_1,x_2\in X$ and $n\ge)$ the inequality

$$d(f_n(x_2),f_n(x_1))\le Md(x_2,x_1).$$

Assume for each $x\in X$ that

$$f(x)=\lim_{n\to\infty} f_n(x)$$

exists. Show f is continuous.

Solution. We have

$$\begin{aligned} d(f(x_2),f(x_1))) &= \lim_{n\to\infty} d(f_n(x_2),f_n(x_1)) \\ &\le \lim_{n\to\infty} Md(x_2,x_1) \\ &= Md(x_2,x_1). \end{aligned}$$

Thus $d(f(x_2),f(x_1)))\le Md(x_2,x_1)$. This shows that f is a Lipschitz function and therefore continuous. □

4. (a) Give our official definition of $\sin(x)$.

Solution. It is defined by the series

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

□

(b) Prove $\sin(1)$ is irrational.

Solution. From the series defining $\sin(1)$ we have

$$\sin(1)=1-\frac{1}{3!}+\frac{1}{5!}-\frac{1}{7!}+\frac{1}{9!}-+\cdots$$

By the alternating series test this implies $\sin(1)$ is positive (and in fact that $1-1/3! < \sin(1) < 1$. Towards a contradiction assume that $\sin(1)$ is rational. Then there are positive integers p and q with

$$\frac{p}{q} = \sin(1)$$

Let n be a odd integer with $2n+1 > q$ with and group the series for $\sin(1)$ as

$$\sin(1) = \frac{p}{q} = \left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots - \frac{1}{(2n+1)!}\right) + R_n$$

where

$$R_n = \frac{1}{(2n+3)!} - \frac{1}{(2n+5)!} + \frac{1}{(2n+7)!} - \cdots$$

Multiply both sides of the equation for $\sin(1)$ by $(2n+1)!$ and rearrange a but to get

$$\frac{p}{q}(2n+1)! - \left(1 - \frac{1}{3!} + \frac{1}{5!} - \cdots - \frac{1}{(2n+1)!}\right)(2n+1)! \quad = (2n+1)!R_n$$

As $2n+1 > q$ the left side of this equation is an integer. On the other hand after doing a bunch of canceling we have

$$\begin{aligned}(2n+1)!R_n =& \frac{1}{(2n+2)(2n+3)} - \frac{1}{(2n+2)(2n+3)(2n+4)(2n+5)} \\ &+ \frac{1}{(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)} - \cdots\end{aligned}$$

and by the alternating series test this is a number strictly between 0 and $1/((2n+2)(2n+3))$ and there it is not an integer. This contradiction completes the proof. □

5. (a) State the ***binomial theorem*** for $(1+x)^\alpha$ where α is a real number.

Solution. For $|x| < 1$ the function $(1+x)^\alpha$ has the convergent series expansion

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

where

$$\binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}.$$

□

(b) Find the power series expansion of

$$F(x) = \int_0^x \sqrt{1+t^3}\, dt$$

about the point $x = 0$.

Solution.

$$\begin{aligned}
F(x) &= \int_0^x \sqrt{1+t^3}\,dt \\
&= \int_0^x (1+t^3)^{\frac{1}{2}}\,dt \\
&= \int_0^x \left(\sum_{k=0}^{\infty}\binom{1/2}{k} t^{3k}\right) dt \\
&= \int_0^x \left(\sum_{k=0}^{\infty}\binom{1/2}{k} t^{3k}\right) dt \\
&= \sum_{k=0}^{\infty}\binom{1/2}{k}\frac{x^{3k+1}}{2k+1}.
\end{aligned}$$

□

6. Let $f\colon[-1,1]\to\mathbf{R}$ be a continuous function and assume for all integers $n\geq 0$ that

$$\int_{-1}^{1} f(x)x^n\,dx = 0.$$

(a) Show for every polynomial $p(x)$ that

$$\int_{-1}^{1} f(x)p(x)\,dx = 0.$$

Solution. Let

$$p(x) = \sum_{k=0}^{n} a_k x^k.$$

Then using the linearity of the integral we find

$$\begin{aligned}
\int_{-1}^{1} f(x)p(x)\,dx &= \int_{-1}^{1} f(x)\Bigl(\sum_{k=0}^{n} a_k x^k\Bigr)\,dx \\
&= \sum_{k=0}^{n} a_k \int_{-1}^{1} f(x)x^k\,dx \\
&= \sum_{k=0}^{n} a_k\cdot 0 \\
&= 0.
\end{aligned}$$

□

(b) By taking a limit of appropriate polynomials show

$$\int_{-1}^{1} f(x)^2\,dx = 0$$

and therefore $f(x)\equiv 0$.

Solution. By the Weierstrass Approximation Theorem there is a sequence of polynomials $p_1(x), p_2(x), p_3(x), \ldots$ with

$$\lim_{n\to\infty} p_n(x) = f(x)$$

uniformly on $[-1, 1]$. But then

$$\lim_{n\to\infty} p_n(x)f(x) = f(x)^2$$

and the convergence is uniform. We know that taking uniform limits pass through integrals. Therefore

$$\begin{aligned}
\int_{-1}^{1} f(x)^2\,dx &= \int_{-1}^{1} \lim_{n\to\infty} p_n(x)f(x)\,dx \\
&= \lim_{n\to 0} \int_{-1}^{1} p_n(x)f(x)\,dx \\
&= \lim_{n\to 0} 0 \\
&= 0.
\end{aligned}$$

The functions $g = f(x)^2$ is continuous and $g \geq 0$ on $[-1, 1]$. We have shown that if g is continuous, non-negative, and $\int_a^b g(x)\,dx = 0$, then $g \equiv 0$. Thus $f^2 \equiv 0$, which implies $f \equiv 0$. $\square$

7. Let $f\colon \mathbf{R} \to \mathbf{R}$ be continuous and $a \in \mathbf{R}$. Define

$$G(x) = \int_{a-x}^{a+x} f(t)\,dt$$

Show that

$$G(0) = 0$$

and that G is differentiable with

$$G'(x) = f(a+x) + f(a-x).$$

Hint: Don't work too hard on this, to show differentiability you just need to do a bit of rewriting and quote the right theorem.

Solution. Write G as

$$G(x) = \int_{a}^{a+x} f(t)\,dt - \int_{a}^{a-x} f(t)\,dt.$$

Then this is differentiable by the Fundamental Theorem of Calculus and the chain rule and

$$\begin{aligned}
G'(x) &= f(a+x)\frac{d}{dx}(a+x) - f(a-x)\frac{d}{dx}(a-x) \\
&= f(a+x) + f(a-x).
\end{aligned}$$

Finally

$$G(0) = \int_{a}^{a+0} f(t)\,dt - \int_{a}^{a-0} f(t)\,dt = 0 + 0 = 0.$$

$\square$