Mathematics 555 Test 3, Take Home Portion.

The problems are 10 points each.

1. If the series of constants $\sum_{k=0}^{\infty} c_k$ converges, show that the series $\sum_{k=0}^{\infty} k^3 2^k c_k x^k$ has radius of convergence at least 1/2.

Solution. As the series $\sum_{k=0}^{\infty} c_k$, the terms are bounded, that is there is a constant B such that

$$|c_k| \le B.$$

Let $|x| < 1/2$. Then

$$\left| k^3 2^k c_k x^k \right| \le Bk^3 (2|x|)^k$$

Using the ratio test on the series

$$\sum_{k=0}^{\infty} Bk^3 (2|x|)^k$$

get

$$\text{Ratio} = \lim_{k\to\infty} \frac{B(k+1)^3(2|x|)^{k+1}}{Bk^3(2|x|)^k} = 2|x| < 1$$

as $|x| < 1/2$. Thus the series $\sum_{k=0}^{\infty} Bk^3(2|x|)^k$ converges and therefore $\sum_{k=0}^{\infty} k^3 2^k c_k x^k$ converges by comparison. This works for all x with $|x| < 1/2$, so the radius of convergence is at least 1/2. $\square$

2. Let $\sum_{k=0}^{\infty} a_k$ be a convergent series. Then the series

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

has radius of convergence at least one (you may assume this).

$$A_n = a_0 + a_1 + \cdots + a_n.$$

Show the series $F(x) = \sum_{n=0}^{\infty} A_n x^n$ also has radius of convergence at least one and that for $|x| < 1$

$$F(x) = \frac{f(x)}{1-x}.$$

Solution. Let $|x| < 1$. Then

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

converges absolutely. We have the expansion

$$\frac{1}{1-x} = \sum_{n=1}^{\infty} x^n$$

which also converges absolutely. Let

$$f(x)\frac{1}{1-x} = \sum_{k=0}^{\infty} c_k$$

be the Cauchy product of these series. The we have a theorem which says that this converges absolutely. By the definition of the Cauchy product

$$c_n = \sum_{k=0}^{n} a_k x^k x^{n-k} = \sum_{k=0}^{n} a_k x^n = A_n x^n.$$

Thus we have

$$F(x) = \sum_{n=0}^{\infty} A_n x^n$$

is the Cauchy product of the series for $f(x)$ and $1/(1-x)$ and so

$$F(x) = \frac{f(x)}{1-x}.$$

This holds for all x with $|x| < 1$ and therefore the radius or convergence for $F(x)$ is at least 1. □

3. Let $a_0, a_1, a_2, \ldots$ be the sequence defined by $a_0 = a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Prove

$$f(x) = \frac{1}{1-x-x^2}.$$

Hint: Start by noting $f(x) = 1 + x + \sum_{n=2}^{\infty} a_n x^n$.

Solution. Using the hint:

$$\begin{aligned}
f(x) &= 1 + x + \sum_{n=2}^{\infty} a_n x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (a_{n-1} + a_{n-2}) x^n \\
&= 1 + x + \sum_{n=2}^{\infty} a_{n-1} x^n + \sum_{n=2}^{\infty} a_{n-1} x^n \\
&= 1 + x + x \sum_{n=2}^{\infty} a_{n-1} x^{n-1} + x^2 \sum_{n=2}^{\infty} a_{n-2} x^{n-2} \\
&= 1 + x + x \sum_{k=1}^{\infty} a_k x^k + x^2 \sum_{k=0}^{\infty} a_k x^k \qquad \text{(change of index in sums.)} \\
&= 1 + x + x(f(x) - 1) + x^2 f(x) \\
&= 1 + (x + x^2) f(x).
\end{aligned}$$

Solving this for $f(x)$ gives

$$f(x) = \frac{1}{1 - x - x^2}$$

as required. □

4. Let $\varphi \colon \mathbf{R} \to \mathbf{R}$ be a continuous function with $\varphi \geq 0$ and

$$\int_{-\infty}^{\infty} \varphi(x)\, dx = 1.$$

Prove

$$K_n(x) = n\varphi(nx)$$

is a Dirac sequence.

Solution. We need to show three things.

(a) $K_n(x) \geq 0$ for all x.

(b)

$$\int_{-\infty}^{\infty} K_n(x)\, dx = 1$$

(c) For all $\delta > 0$

$$\lim_{n\to\infty} \int_{|x|>\delta} K_n(x)\, dx = 0.$$

The first of these is clear as $\varphi(x) \geq 0$ and therefore $K_n(x) = n\varphi(nx) \geq 0$.

For (b) we do the change of variable $u = nx$ (so that $du = ndx$) to get

$$\int_{-\infty}^{\infty} K_n(x)\, dx = \int_{-\infty}^{\infty} n\varphi(nx)\, dx = \int_{-\infty}^{\infty} \varphi(u)\, du = 1.$$

For (c) we do the same change of variable to get

$$\begin{aligned}\int_{|x|\geq\delta} K_n(x)\,dx &= \int_{|x|\geq\delta} n\varphi(nx)\,dx \\ &= \int_{|u|\geq n\delta} \varphi(u)\,du \\ &= \int_{-\infty}^{\infty} \varphi(u)\,du - \int_{-n\delta}^{n\delta} \varphi(u)\,du\end{aligned}$$

and therefore, by the definition of the improper integral $\int_{-\infty}^{\infty} \varphi(u)\,du$

$$\begin{aligned}\lim_{n\to\infty} \int_{|x|\geq\delta} K_n(x)\,dx &= \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \varphi(u)\,du - \int_{-n\delta}^{n\delta} \varphi(u)\,du \right) \\ &= \int_{-\infty}^{\infty} \varphi(u)\,du - \int_{-\infty}^{\infty} \varphi(u)\,du \\ &= 0.\end{aligned}$$

This finishes the proof. □

5. (This one is a somewhat more involved than the other problems.) Show that the function

$$f(x) = \begin{cases} 0, & x \leq 0; \\ e^{-1/x}, & x > 0. \end{cases}$$

has continuous derivatives of all orders and

$$f^{(k)}(0) = 0$$

for all $k = 0, 1, 2, 3, \ldots$. Therefore the Taylor series for $f(x)$ at $x = 0$ is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{0}{k!} x^k = 0.$$

Thus for $x > 0$ the series converges, but does not converge to $f(x) = e^{-1/x}$. *Hint:* A good place to start, is to show for all $n \geq 0$ that

$$\lim_{x \searrow 0} \frac{e^{-1/x}}{x^n} = 0.$$

Solution. We start by looking at the derivatives of $f(x)$ for $x > 0$. The first several are

$$\begin{aligned}
f(x) &= e^{-1/x} \\
f'(x) &= \left(\frac{1}{x^2}\right) e^{-1/x} \\
f''(x) &= \left(\frac{-2x+1}{x^4}\right) e^{-1/x} \\
f'''(x) &= \left(\frac{6x^2-6x+1}{x^6}\right) e^{-1/x} \\
f^{(4)}(x) &= \left(\frac{-24x^3+36x^2-12x+1)}{x^8}\right) e^{-1/x} \\
f^{(5)}(x) &= \left(\frac{120x^4-240x^3+120x^2-20x+1}{x^{10}}\right) e^{-1/x}
\end{aligned}$$

which leads to the conjecture that

$$f^{(n)} = \left(\frac{p_n(x)}{x^{2n}}\right) e^{-1/x}$$

where p_n is a polynomial of degree $n-1$. Assuming this holds for n we find the next derivative is

$$f^{(n+1)}(x) = \left(\frac{x^2 p_n'(x) - 2nxp_n(x) + p_n(x)}{x^{2n+2}}\right) e^{-1/x^2}$$

and

$$p_{n+1}(x) = x^2 p_n'(x) - 2nxp_n(x) + p_n(x)$$

is a polynomial of degree n. So the conjecture follows by induction. We record this as

Lemma 1. *The function $f(x)$ has derivatives of all order on the intervals $(-\infty, 0)$ and $(0, \infty)$ and*

$$f^{(n)}(x) = \begin{cases} \left(\dfrac{p_n(x)}{x^{2n}}\right) e^{-1/x}, & x > 0; \\ 0, & x < 0. \end{cases}$$

□

Lemma 2. *For any integer m the limit*

$$\lim_{x \searrow 0} \frac{e^{-1/x}}{x^m} = 0$$

holds.

Proof. We do the change of variable $y = 1/x$ in the limit and use that exponential grow faster than polynomials.

$$\begin{aligned}\lim_{x \searrow 0} \frac{e^{-1/x}}{x^m} &= \lim_{y \to \infty} \frac{e^{-y}}{(1/y)^m} \\ &= \lim_{y \to \infty} \frac{y^m}{e^y} \\ &= 0.\end{aligned}$$

□

Lemma 3. *If $p(x)$ is a polynomial and m is an integer, then*

$$\lim_{x \searrow 0} \frac{p(x)}{x^m} e^{-1/x} = 0.$$

Proof. By the previous lemma:

$$\lim_{x \searrow 0} \frac{p(x)}{x^m} e^{-1/x} = \lim_{x \searrow 0} p(x) \lim_{x \searrow 0} \frac{e^{-1/x}}{x^m} = p(0) \cdot 0 = 0.$$

□

Lemma 4. *Let $p(x)$ be a polynomial and m an integer. Let*

$$g(x) = \begin{cases} \dfrac{p(x)}{x^m} e^{-1/x}, & x > 0; \\ 0, & x \geq 0. \end{cases}$$

Then $g(x)$ is differentiable and

$$g'(x) = \begin{cases} \left(\dfrac{x^2 p'(x) - mxp(x) + p(x)}{x^{m+2}} \right) e^{-1/x}, & x > 0; \\ 0, & x < 0. \end{cases}$$

Proof. For $x \neq 0$ this is clear, as the constant function 0 is differentiable on $(-\infty, 0)$ and $p(x)e^{-1/x}/x^m$ is differentiable on $(0, \infty)$ and has the indicated derivative. All that remains is to show g is differentiable at 0 and $g'(x) = 0$. By definition

$$g'(0) = \lim_{x \to 0} \frac{g(x)}{x}.$$

We split this into two one sided limits.

$$lim_{x \nearrow 0} \frac{g(x)}{x} = \lim_{x \nearrow 0} \frac{0}{x} = 0.$$

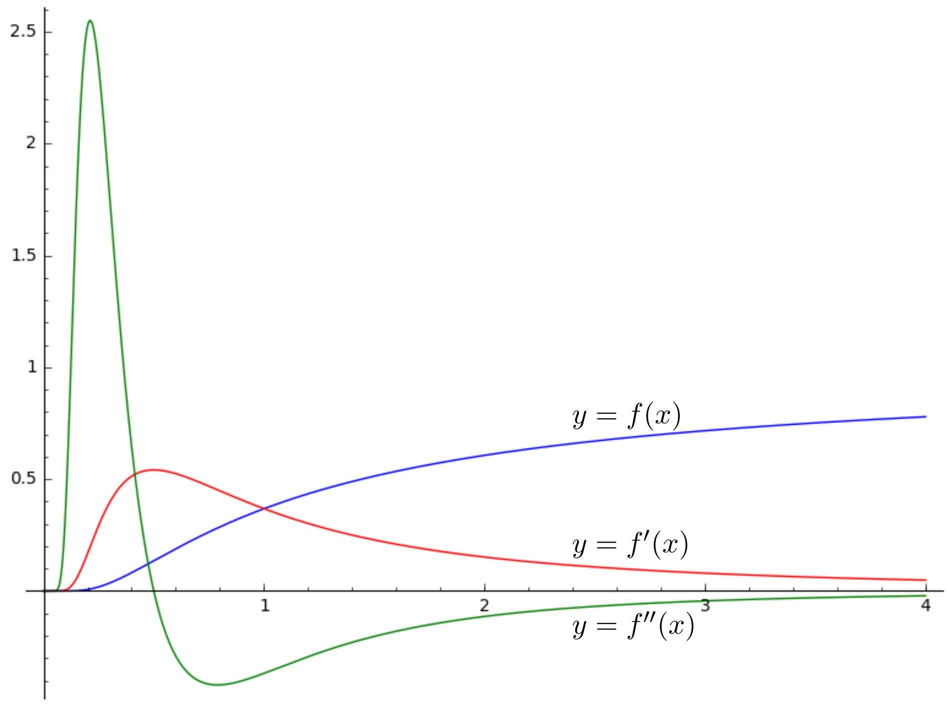
And

$$\lim_{x \searrow 0} \frac{g(x)}{x} = \lim_{x \searrow 0} \frac{p(x)}{x^{m+1}} e^{-1/x} = 0$$

by Lemma 3. This completes the proof of the lemma. □

Returning to the problem. That $f(x)$ has derivatives of all orders for $x \neq 0$ follows from Lemma 1. That it has f has derivatives of all orders at $x = 0$ and $f^{(n)}(0) = 0$, follows from Lemma 1, Lemma 4, and induction.

In case you are curious here are the graphs of f and its first two derivatives.



□