

## CONTENTS

1.	Riemann Integration	1
2.	The Fundamental Theorem of Calculus.	7
3.	Definition of the logarithm and exponential functions.	14

### 1. RIEMANN INTEGRATION

We start with some definitions.

**Definition 1.** Let  $[a, b]$  be a closed bounded interval. Then a **partition** of  $[a, b]$  is a list of points  $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ . We denote it by  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$ . We also use the notation

$$\Delta x_j = x_j - x_{j-1}.$$

(See Figure 1.)

□

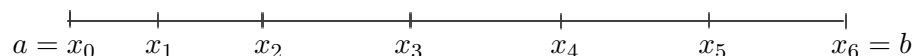


FIGURE 1. A partition of the interval  $[a, b]$  into  $n = 6$  pieces.

The  $j$ -th interval  $[x_{j-1}, x_j]$  has length  $\Delta x_j = x_j - x_{j-1}$ .

**Definition 2.** The function  $\varphi$  is a **step function** on  $[a, b]$  so that there is a partition  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  of  $[a, b]$  and  $\varphi$  is constant on each open interval  $(x_{j-1}, x_j)$  for  $j = 1, 2, \dots, n$ . We denote the set of all step functions on  $[a, b]$  by  $\mathcal{S}[a, b]$ .

□

**Proposition 3.** The set  $\mathcal{S}[a, b]$  is a vector space. That is if  $\varphi_1, \varphi_2 \in \mathcal{S}[a, b]$  and  $c_1, c_2 \in \mathbb{R}$ , then  $c_1\varphi_1 + c_2\varphi_2 \in \mathcal{S}[a, b]$ .

**Problem 1.** Prove this.

□

**Proposition 4.** If  $f$  is a bounded function on the closed bounded interval  $[a, b]$  then  $f$  is integrable if and only if all  $\varepsilon > 0$  there are step functions  $\varphi, \psi \in \mathcal{S}[a, b]$  such that

$$\varphi \leq f \leq \psi$$

and

$$\int_a^b (\psi - \varphi) dx < \varepsilon.$$

**Problem 2.** Prove this. *Hint:* We outlined the proof in class.

□

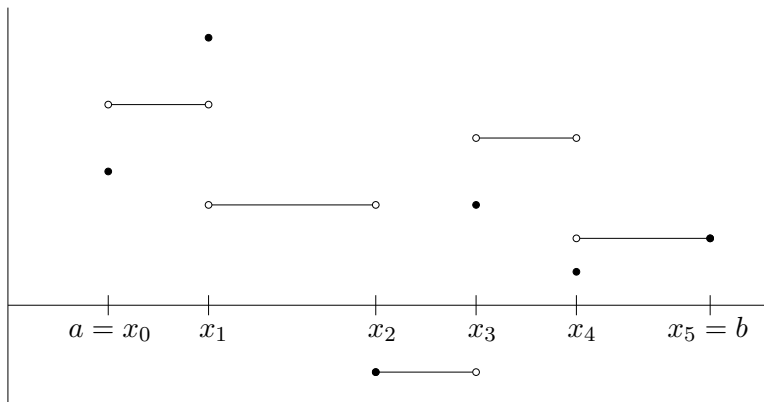


FIGURE 2. A step function,  $\varphi$ , for the interval  $[a, b]$  partitioned into five subintervals. By definition  $\varphi$  is constant on each of the open intervals  $(x_{j-1}, x_j)$  for  $j = 1, 2, 3, 4, 5$ . No assumption is made about the values at the points  $x_j$ .

**Definition 5.** Let  $\varphi \in \mathcal{S}[a, b]$  and let  $\mathcal{P} = \{a = x_0, x_1, \dots, x_n = b\}$  be a partition such that  $\varphi$  has the constant value  $c_j$  on the open interval  $(x_{j-1}, x_j)$ . Then the *integral* of  $\varphi$  on the interval  $[a, b]$

$$\int_a^b \varphi(x) dx = \sum_{j=1}^n c_j (x_j - x_{j-1}). \quad \square$$

Setting  $\Delta x_j = x_j - x_{j-1}$  (the length of the interval  $(x_{j-1}, x_j)$ ) this can also be written as

$$\int_a^b \varphi(x) dx = \sum_{j=1}^n c_j \Delta x_j.$$

When all the  $c_j$ 's are positive, this is just the area under the graph of  $\varphi$  computed by adding up the area rectangles. In the general case (where the  $c_j$ 's can be negative) this is the area where the area below the graph  $x$ -axis counted as negative.

**Proposition 6.** The integral is linear on  $\mathcal{S}[a, b]$ . That is if  $\varphi_1, \varphi_2 \in \mathcal{S}[a, b]$ , and  $c_1, c_2 \in \mathbb{R}$ , then

$$\int_a^b (c_1 \varphi_1 + c_2 \varphi_2) dx = c_1 \int_a^b \varphi_1(x) dx + c_2 \int_a^b \varphi_2(x) dx. \quad \square$$

**Problem 3.** Prove this.  $\square$

**Definition 7.** Let  $f: [a, b] \rightarrow \mathbb{R}$  be a bounded function. Then define the *upper integral* of  $f$  to be

$$\overline{\int_a^b} f(x) dx = \inf \left\{ \int_a^b \psi(x) dx : \psi \in \mathcal{S}[a, b] \text{ and } f(x) \leq \psi(x) \text{ all } x \in [a, b] \right\}.$$

Likewise the **lower integral** of  $f$  is

$$\int_a^b f(x) dx = \inf \left\{ \int_a^b \psi(x) dx : \psi \in \mathcal{S}[a, b] \text{ and } f(x) \geq \psi(x) \text{ all } x \in [a, b] \right\}.$$

**Definition 8.** The bounded function  $f: [a, b] \rightarrow \mathbb{R}$  is **Riemann integrable** if and only if

$$\int_a^b f(x) dx = \overline{\int}_a^b f(x) dx.$$

In this case the **integral** of  $f$  is the common value of the upper and lower integral

$$\int_a^b f(x) dx = \int_a^b f(x) dx = \overline{\int}_a^b f(x) dx.$$

We denote the set of all Riemann integrable functions on  $[a, b]$  by  $\mathcal{R}[a, b]$ .

The following gives a method for showing that a function is Riemann integrable without having to compute the upper and lower integrals.

**Theorem 9.** *The bounded function  $f: [a, b] \rightarrow \mathbb{R}$  is Riemann integrable if and only if for all  $\varepsilon > 0$  there are step functions  $\varphi, \psi \in \mathcal{S}[a, b]$  with*

$$\varphi \leq f \leq \psi$$

on  $[a, b]$  and

$$\int_a^b (\psi(x) - \varphi(x)) dx < \varepsilon.$$

*Proof.* Done in class. □

**Theorem 10.** *The set  $\mathcal{R}[a, b]$  is a vector space and the integral is linear on  $\mathcal{R}[a, b]$ .*

*Proof.* This was also done in class. □

If  $f$  is a monotone increasing function on  $[a, b]$  and  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  is a partition of  $[a, b]$  define two step functions by  $\varphi_{f, \mathcal{P}}(b) = f(b)$ ,

$$\varphi_{f, \mathcal{P}}(x) = f(x_{j-1}) \quad \text{for} \quad x \in [x_{j-1}, x_j)$$

and  $\psi_{f, \mathcal{P}}(b) = f(b)$

$$\psi_{f, \mathcal{P}}(x) = f(x_j) \quad \text{for} \quad x \in [x_{j-1}, x_j].$$

See Figure 3

**Proposition 11.** *If  $f$  is monotone increasing on  $[a, b]$  then for any partition,  $\mathcal{P}$ , of  $[a, b]$ , with the notation above,*

$$\varphi_{f, \mathcal{P}} \leq f \leq \psi_{f, \mathcal{P}}$$

on  $[a, b]$ .

**Problem 4.** Prove this. □

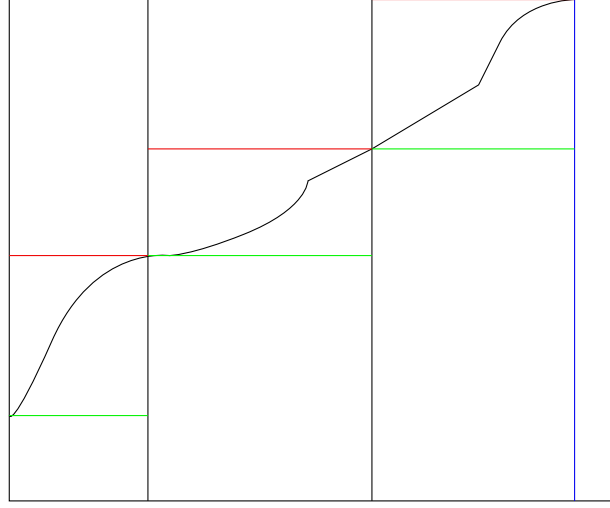


FIGURE 3. A monotone increasing function on  $[a, b]$  and a partition,  $\mathcal{P}$ , with  $n = 3$  showing the lower step function  $\varphi_{f,\mathcal{P}}$  (in green) and the upper step function  $\psi_{f,\mathcal{P}}$  (in red).

**Definition 12.** Given a positive integer  $n$  and a closed bounded interval  $[a, b]$  the *uniform partition* of  $[a, b]$  into  $n$  sub-intervals is the partition  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  with

$$x_j = a + j \left( \frac{b-a}{n} \right)$$

for  $j = 0, 1, \dots, n$ . Note in this case all the lengths,  $\Delta x_j$  of the sub-intervals  $[x_{j-1}, x_j]$  have the same value  $\Delta x = \Delta x_j = (b-a)/n$ .  $\square$

Now let us consider the monotone increasing function  $f$  on the interval  $[a, b]$  with the uniform partition,  $\mathcal{P}$ , of  $[a, b]$  with  $n = 4$ . Then  $\Delta x = \Delta x_j = (b-a)/4$  and  $\varphi_{f,\mathcal{P}} \leq f \leq \psi_{f,\mathcal{P}}$ . Also

$$\int_a^b \varphi_{f,\mathcal{P}}(x) dx = (f(x_0) + f(x_1) + f(x_2) + f(x_3)) \Delta x$$

and

$$\int_a^b \psi_{f,\mathcal{P}}(x) dx = (f(x_1) + f(x_2) + f(x_3) + f(x_4)) \Delta x.$$

Thus

$$\int_a^b (\psi_{f,\mathcal{P}}(x) - \varphi_{f,\mathcal{P}}(x)) dx = (f(x_4) - f(x_0)) \Delta x = (f(b) - f(a)) \Delta x$$

There is nothing special about  $n = 4$  in this:

**Problem 5.** Show that if  $f$  is monotone increasing on  $[a, b]$ ,  $n$  is a positive integer and  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  is the uniform partition of  $[a, b]$  into  $n$

sub-intervals, then, with the notation above,

$$\int_a^b (\psi_{f,\mathcal{P}}(x) - \varphi_{f,\mathcal{P}}(x)) dx = (f(b) - f(a)) \Delta x = \frac{(f(b) - f(a))(b - a)}{n}. \quad \square$$

**Theorem 13.** *If  $f$  is a monotone function on the closed bounded interval  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ .*

**Problem 6.** Prove this. *Hint:* With out loss of generality assume  $f$  is monotone increasing (if  $f$  is monotone decreasing replace  $f$  by  $-f$ ). Let  $\varepsilon > 0$  and let  $n$  be a positive integer such that

$$\frac{(f(b) - f(a))(b - a)}{n} < \varepsilon$$

and use Proposition 4 and the last problem.  $\square$

**Theorem 14.** *Let  $f$  be a continuous function on  $[a, b]$ . Then  $f$  is integrable on  $[a, b]$ .*

*Proof.* Let  $\varepsilon > 0$ . As  $f$  is continuous on the closed bounded set  $[a, b]$  it is uniformly continuous on  $[a, b]$ . Thus there is an  $\delta > 0$  such that for  $x, y \in [a, b]$ .

$$|x - y| < \delta \implies |f(x) - f(y)| < \frac{\varepsilon}{b - a}.$$

Let  $n$  be a positive integer such that

$$\frac{b - a}{n} = \Delta x < \delta$$

and let  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  be the uniform partition of  $[a, b]$  into  $n$  sub-intervals. Set

$$m_j = \inf\{f(x) : x \in [x_{j-1}, x_j]\} = \min\{f(x) : x \in [x_{j-1}, x_j]\},$$

$$M_j = \sup\{f(x) : x \in [x_{j-1}, x_j]\} = \max\{f(x) : x \in [x_{j-1}, x_j]\}$$

where the infimum is achieved as a minimum and the supremum is achieved as a maximum because continuous functions on closed bounded sets achieve their maximums and minimums. Define step functions  $\varphi$  and  $\psi$  on  $[a, b]$   $\varphi(b) = \psi(b) = f(b)$  and

$$\begin{aligned} \varphi(x) &= m_j & \text{for } x_{j-1} \leq x < x_j \\ \psi(x) &= M_j & \text{for } x_{j-1} \leq x < x_j. \end{aligned}$$

Then

$$\varphi \leq f \leq \psi$$

and

$$\int_a^b (\varphi(x) - \psi(x)) dx = \sum_{j=1}^n (M_j - m_j) \left( \frac{b - a}{n} \right).$$

As  $f$  is continuous on the closed bounded interval  $[x_{j-1}, x_j]$ ,  $f$  achieves its maximum and minimum on this interval. Thus there are  $\alpha_j, \beta_j \in [x_{j-1}, x_j]$

with  $f(\alpha_j) = m_j$  and  $f(\beta_j) = M_j$ . But then  $|\alpha_j - \beta_j| \leq \Delta x < \delta$  and therefore

$$M_j - m_j = |f(\beta_j) - f(\alpha_j)| < \frac{\varepsilon}{b-a}.$$

Thus

$$\int_a^b (\varphi(x) - \psi(x)) dx = \sum_{j=1}^n (M_j - m_j) \left( \frac{b-a}{n} \right) < \sum_{j=1}^n \frac{\varepsilon}{b-a} \left( \frac{b-a}{n} \right) = \varepsilon$$

and the result now follows from Proposition 4.  $\square$

**Lemma 15.** *Let  $\alpha, \beta \in \mathbb{R}$ , then*

$$|\max\{\alpha, 0\} - \max\{\beta, 0\}| \leq |\alpha - \beta|.$$

**Problem 7.** Prove this by splitting it into the four cases (i)  $\alpha, \beta \geq 0$ , (ii)  $\alpha \geq 0, \beta < 0$ , (iii)  $\alpha < 0, \beta \geq 0$ , and (iv)  $\alpha, \beta < 0$ . This is not to be handed in.  $\square$

**Proposition 16.** *If  $f \in \mathcal{R}[a, b]$  then so is  $g = \max\{f, 0\}$ .*

*Proof.* Let  $\varepsilon > 0$ . Let  $\varphi$  and  $\psi$  be step functions on  $[a, b]$  such that  $\varphi \leq f \leq \psi$  and  $\int_a^b (\psi - \varphi) dx < \varepsilon$ . Then

$$\varphi_0 = \max\{0, \varphi\}, \quad \psi_0 = \max\{0, \psi\}$$

are step functions,  $\varphi_0 \leq \max\{f, 0\} \leq \psi_0$  and  $0 \leq \psi_0 - \varphi_0 \leq \psi - \varphi$ . Thus, using Lemma 15,

$$\int_a^b (\psi_0 - \varphi_0) dx \leq \int_a^b (\psi - \varphi) dx < \varepsilon$$

and so  $\max\{f, 0\}$  is integrable by Proposition 4.  $\square$

This implies a good deal more because of the following elementary result.

**Lemma 17.** *For real numbers  $a, b$  the following hold*

$$\begin{aligned} \min\{a, 0\} &= -\max\{-a, 0\}, \\ |a| &= \max\{a, 0\} + \max\{-a, 0\}, \\ \max\{a, b\} &= a + \max\{0, b - a\}, \\ \min\{a, b\} &= a + \min\{0, b - a\}. \end{aligned}$$

*Proof.* Left to reader (and you don't have to turn these in). We did enough of this type of thing last term that I believe you can do it.  $\square$

**Proposition 18.** *If  $f$  and  $g$  are integrable on  $[a, b]$  then so are  $|f|$ ,  $\min\{f, g\}$  and  $\max\{f, g\}$ .*

*Proof.* This follows easily from Proposition 16 and Lemma 17.  $\square$

**Lemma 19.** *If  $f$  is integrable on  $[a, b]$  then so is  $f^2$ .*

**Problem 8.** Prove this. *Hint:* As  $f^2 = |f|^2$  and  $|f|$  is also integrable by replacing  $f$  by  $|f|$  we can assume  $f \geq 0$ . As  $f$  is integrable it is bounded, say  $0 \leq f \leq B$  on  $[a, b]$ . Also as  $f$  is integrable on  $[a, b]$  for  $\varepsilon > 0$  there are step functions  $\varphi, \psi$  such that

$$\varphi \leq f \leq \psi$$

and

$$\int_a^b (\psi - \varphi) dx < \frac{\varepsilon}{2B}.$$

By replacing  $\varphi$  by  $\max\{0, \varphi\}$  and  $\psi$  by  $\min\{\psi, B\}$  we can assume  $0 \leq \varphi$  and  $\psi \leq B$ . Then  $\varphi^2$  and  $\psi^2$  are step functions and

$$\varphi^2 \leq f^2 \leq \psi^2$$

and

$$0 \leq \psi^2 - \varphi^2 = (\psi + \varphi)(\psi - \varphi) \leq (\psi + \psi)(\psi - \varphi) \leq (B + B)(\psi - \varphi).$$

You should now be able to show

$$\int_a^b (\psi^2 - \varphi^2) dx < \varepsilon$$

so that Proposition 4 applies.  $\square$

**Proposition 20.** If  $f$  and  $g$  are integrable on  $[a, b]$  then so is the product  $fg$ .

**Problem 9.** Prove this. *Hint:* Show

$$fg = \frac{(f+g)^2 - (f-g)^2}{4}$$

and use Lemma 19.  $\square$

## 2. THE FUNDAMENTAL THEOREM OF CALCULUS.

**Proposition 21.** If  $a < b < c$  and  $f$  is integrable on  $[a, c]$  then the restrictions  $f|_{[a, b]}$  and  $f|_{[b, c]}$  are integrable on  $[a, b]$  and  $[b, c]$  respectively and

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$

*Proof.* We have shown in class that for any bounded function on  $[a, c]$  that

$$\begin{aligned} \overline{\int_a^c f(x)} dx &= \overline{\int_a^b f(x)} dx + \overline{\int_b^c f(x)} dx, \\ \underline{\int_a^c f(x)} dx &= \underline{\int_a^b f(x)} dx + \underline{\int_b^c f(x)} dx. \end{aligned}$$

As  $f$  is integrable on  $[a, c]$

$$\begin{aligned}
 \int_a^c f(x) dx &= \overline{\int}_a^c f(x) dx \\
 &= \overline{\int}_{\underline{a}}^c f(x) dx \\
 &= \overline{\int}_{\underline{a}}^b f(x) dx + \overline{\int}_{\underline{b}}^c f(x) dx \\
 &\leq \overline{\int}_{\underline{a}}^b f(x) dx + \overline{\int}_b^c f(x) dx \\
 &= \overline{\int}_{\underline{a}}^c f(x) dx \\
 &= \int_a^c f(x) dx.
 \end{aligned}$$

Thus equality must hold at all the intermediate inequalities. Therefore

$$\overline{\int}_{\underline{a}}^b f(x) dx = \overline{\int}_a^b f(x) dx \quad \text{and} \quad \overline{\int}_{\underline{b}}^c f(x) dx = \overline{\int}_b^c f(x) dx$$

which implies the restrictions  $f|_{[a,b]}$  and  $f|_{[b,c]}$  are integrable. The rest follows from

$$\int_a^b f(x) dx = \overline{\int}_a^b f(x) dx \quad \text{and} \quad \int_b^c f(x) dx = \overline{\int}_b^c f(x) dx$$

and that equality holds in the displayed inequality.  $\square$

**Proposition 22.** Let  $f$  be integrable on  $[a, b]$  and let  $[\alpha, \beta] \subseteq [a, b]$ . The  $f$  is integrable on  $[\alpha, \beta]$ .

**Problem 10.** Prove this. *Hint:*  $[\alpha, \beta] = [a, \beta] \cap [\alpha, b]$  and Proposition 21.  $\square$

It is useful to define  $\int_a^b f(x) dx$  even in the cases where  $a = b$  and  $b < a$ .

**Definition 23.** For any function  $f$  define

$$\int_a^b f(x) dx = 0.$$

If  $b < a$  and  $f$  is integrable on  $[b, a]$  define

$$\int_a^b f(x) dx = - \int_b^a f(x) dx. \quad \square$$

**Proposition 24.** If  $f$  is integrable on the interval  $[x_1, x_2]$  and  $a, b, c \in [x_1, x_2]$  then, with the definitions above,

$$\int_a^c f(x) dx = \int_a^b f(x) dx + \int_b^c f(x) dx.$$



*Proof.* This is just checking case by case (i.e.  $a \leq b \leq c$ ,  $a \leq c \leq b$  etc.) and is left to the reader. And please do not hand it in.  $\square$

**Proposition 25.** Let  $f(x)$  be integrable on  $[a, b]$  and let  $F: [a, b] \rightarrow \mathbb{R}$  be defined by

$$F(x) = \int_a^x f(t) dt$$

then  $F$  is Lipschitz. That is there is a constant  $M$  such that for all  $x_1, x_2 \in [a, b]$ ,

$$|F(x_2) - F(x_1)| \leq M|x_2 - x_1|$$

and therefore  $F$  is continuous on  $[a, b]$ .

**Problem 11.** Prove this. *Hint:* As  $f$  is integrable on  $[a, b]$ , it is bounded on  $[a, b]$ , say  $|f(x)| \leq M$  on  $[a, b]$ . Without loss of generality we can assume that  $x_1 \leq x_2$ . Then

$$|F(x_2) - F(x_1)| = \left| \int_a^{x_2} f(t) dt - \int_a^{x_1} f(t) dt \right| = \left| \int_{x_1}^{x_2} f(t) dt \right| \leq \int_{x_1}^{x_2} |f(t)| dt$$

and it should be easy from here.  $\square$

**Theorem 26** (Fundamental Theorem of Calculus Form 1). Let  $f$  be integrable on  $[a, b]$ . Define new function  $F: [a, b] \rightarrow \mathbb{R}$  by

$$F(x) = \int_a^x f(t) dt.$$

If  $f$  is continuous at the point  $x \in (a, b)$ , then the derivative of  $F$  exists at  $x$  and

$$F'(x) = f(x).$$

**Problem 12.** Prove this. *Hint:* First note

$$1 = \frac{1}{h} \int_x^{x+h} 1 dt.$$

Multiply by  $f(x)$  to get

$$f(x) = \frac{1}{h} \int_x^{x+h} f(x) dt$$

Also note

$$F(x+h) - F(x) = \int_a^{x+h} f(t) dt - \int_a^x f(t) dt = \int_x^{x+h} f(t) dt.$$

Combining some of these formulas we get

$$\begin{aligned} \frac{F(x+h) - F(x)}{h} - f(x) &= \frac{1}{h} \int_x^{x+h} f(t) dt - \frac{1}{h} \int_x^{x+h} f(x) dt \\ &= \frac{1}{h} \int_x^{x+h} (f(t) - f(x)) dt. \end{aligned}$$

Let  $\varepsilon > 0$ . As  $f$  is continuous at  $x$  there is a  $\delta > 0$  such that

$$|t - x| < \delta \implies |f(t) - f(x)| < \varepsilon.$$

Put this all together to show

$$|h| < \delta \implies \left| \frac{F(x+h) - F(x)}{h} - f(x) \right| < \varepsilon$$

and explain why this shows  $F'(x) = f(x)$ .  $\square$

**Theorem 27** (Fundamental Theorem of Calculus Form 2). *Let  $f$  be continuous on  $[a, b]$  and let  $F$  be continuous on  $[a, b]$  and differentiable  $(a, b)$  with  $F' = f$  on  $(a, b)$ . Then*

$$\int_a^b f(t) dt = F(b) - F(a) = F \Big|_a^b.$$

**Problem 13.** Prove this. *Hint:* Let

$$G(x) = \int_a^x f(t) dt - F(x)$$

and show  $G'(x) = 0$  for  $x \in (a, b)$ .  $\square$

**Corollary 28.** *If  $f$  is continuous on  $[a, b]$  and  $F$  is any anti-derivative of  $f$  on  $[a, b]$  (that is  $F'(x) = f(x)$  for  $x \in [a, b]$ ), then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

**Problem 14.** Prove this.  $\square$

**Definition 29.** Let  $f$  be integrable on  $[a, b]$ . Then the **average value** of  $f$  on  $[a, b]$  is

$$\frac{1}{b-a} \int_a^b f(x) dx. \quad \square$$

**Theorem 30** (The First Mean Value Theorem for Integrals). *If  $f$  is continuous on  $[a, b]$ , then it achieves its average value. That is there is a  $\xi \in (a, b)$  with*

$$f(\xi) = \frac{1}{b-a} \int_a^b f(x) dx.$$

**Problem 15.** Prove this. *Hint:* As  $f$  is continuous on the closed bounded set  $[a, b]$ , it achieves its maximum and minimum on this interval. Let  $m = \min\{f(x) : x \in [a, b]\}$  and  $M = \max\{f(x) : x \in [a, b]\}$  and let  $\alpha, \beta \in [a, b]$  such that  $f(\alpha) = m$  and  $f(\beta) = M$ . Now

$$f(\alpha) = m = \frac{1}{b-a} \int_a^b m dx \leq \frac{1}{b-a} \int_a^b f(x) dx$$

and

$$f(\beta) = M = \frac{1}{b-a} \int_a^b M dx \geq \frac{1}{b-a} \int_a^b f(x) dx$$

and recall the intermediate value theorem.  $\square$

We now prove a somewhat stronger version of the second form of the Fundamental Theorem of Calculus.

**Theorem 31.** *Let  $F$  be continuous on  $[a, b]$  assume that  $F$  is differentiable on  $(a, b)$  and let*

$$f(x) = F'(x)$$

*on  $[a, b]$ . Then*

$$\int_a^b f(x) dx = F(b) - F(a).$$

*(This differs from Theorem 27 as we are only assuming that  $f$  is integrable rather than continuous.)*

*Proof.* Let  $\varepsilon > 0$ . As  $f$  is integrable there are step functions  $\varphi$  and  $\psi$  on  $[a, b]$  with

$$(1) \quad \varphi \leq f \leq \psi \quad \text{and} \quad \int_a^b f dx - \varepsilon \leq \int_a^b \varphi dx \leq \int_a^b \psi dx \leq \int_a^b f dx + \varepsilon.$$

We can assume there is a partition  $\mathcal{P} = \{x_0, x_1, \dots, x_n\}$  such that if  $I_j = [x_{j-1}, x_j]$  then

$$\varphi = \sum_{j=1}^n m_j \chi_{I_j}, \quad \psi = \sum_{j=1}^n M_j \chi_{I_j}.$$

We write  $F(b) - F(a)$  as a telescoping sum:

$$F(b) - F(a) = F(x_n) - F(x_0) = \sum_{j=1}^n (F(x_j) - F(x_{j-1}))$$

As  $F$  is differentiable on  $[x_{j-1}, x_j]$  we can apply the mean value theorem to get that there is a  $\xi_j \in (x_{j-1}, x_j)$  with

$$F(x_j) - F(x_{j-1}) = F'(\xi_j)(x_j - x_{j-1}) = f(\xi_j)(x_j - x_{j-1}) = f(\xi_j)|I_j|.$$

Combining these equations gives

$$F(b) - F(a) = \sum_{j=1}^n (F(x_j) - F(x_{j-1})) = \sum_{j=1}^n f(\xi_j)|I_j|.$$

But  $\varphi \leq f \leq \psi$  which implies  $m_j \leq f(\xi_j) \leq M_j$  and thus

$$\int_a^b \varphi dx = \sum_{j=1}^n m_j |I_j| \leq F(b) - F(a) = \sum_{j=1}^n f(\xi_j) |I_j| \leq \sum_{j=1}^n M_j |I_j| = \int_a^b \psi dx.$$

Combining this with the inequalities (1) gives

$$\int_a^b f dx - \varepsilon \leq F(b) - F(a) \leq \int_a^b f dx + \varepsilon.$$

As  $\varepsilon > 0$  was arbitrary this gives  $F(b) - F(a) = \int_a^b f dx$  as required.  $\square$

**Problem 16.** To see that Theorem 31 really is stronger than Theorem 27 we need to show that there is a function  $F$  on an interval  $[a, b]$  such that  $f = F'$  exists and is integrable on  $(a, b)$  but with  $f$  not continuous on  $(a, b)$ . Let

$$F(x) = \begin{cases} x^2 \cos(1/x), & x \neq 0; \\ 0, & x = 0 \end{cases}$$

Show that  $F$  is differentiable at all points of  $\mathbb{R}$ , and  $f = F'$  is bounded on  $[-1, 1]$ , but  $f$  is not continuous at  $x = 0$ . As  $f$  is continuous at all points other than 0 it is integrable on  $[-1, 1]$ .  $\square$

We can now give the familiar integration by parts formula.

**Theorem 32** (Integration by Parts). *Let  $u$  and  $v$  continuous on  $[a, b]$ , differentiable on  $(a, b)$ , with  $u'$  and  $v'$  integrable on  $[a, b]$ . Then*

$$\int_a^b u(x)v'(x) dx = u(x)v(x) \Big|_{x=a}^b - \int_a^b u'(x)v(x) dx.$$

**Problem 17.** Prove this. *Hint:* This follows from the product rule and the Fundamental Theorem of Calculus in the form

$$\int_a^b (u(x)v(x))' dx = u(x)v(x) \Big|_{x=a}^b.$$

You do have to worry a bit about if the integrals involved exist. Theorem 20 should help here.  $\square$

We now use integration by parts to give another form of the remainder in Taylor's Theorem.

**Lemma 33.** *Let  $f$  be  $k+1$  times differentiable on an open interval  $(\alpha, \beta)$  and assume that  $f^{(k+1)}$  is integrable. Then for  $a, x \in (\alpha, \beta)$  we have*

$$\int_a^x \frac{(x-t)^{k-1}}{(k-1)!} f^{(k)}(t) dt = \frac{f^{(k)}(a)}{k!} (x-a)^k + \int_a^x \frac{(x-t)^k}{k!} f^{(k+1)}(t) dt.$$

**Problem 18.** Prove this. *Hint:* Use integration by parts with  $v'(t) = \frac{(x-t)^{k-1}}{(k-1)!}$  and  $u = f^{(k)}(t)$ .  $\square$

**Theorem 34** (Taylor's Theorem with Integral form of the Remainder). *Let  $f$  be  $n+1$  times differentiable on  $(\alpha, \beta)$  and assume that  $f^{(n+1)}$  is integrable. Then for  $a, x \in (\alpha, \beta)$*

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x)$$

where the remainder term  $R_n(x)$  is given by

$$R_n(x) = \int_a^x \frac{(x-t)^n}{n!} f^{(n+1)}(t) dt.$$

**Problem 19.** Prove this. *Hint:* Note that Lemma 33 can be rewritten as

$$R_{k-1}(x) = \frac{f^{(k)}(a)}{k!}(x-a)^k + R_k(x)$$

and by the Fundamental Theorem of Calculus and integration by parts

$$\begin{aligned} f(x) - f(a) &= \int_a^x f'(t) dt \\ &= - \int_a^x (-1)f'(t) dt \\ &= - \int_a^x \left( \frac{d}{dt}(x-t) \right) f'(t) dt \\ &= - \frac{d}{dt}(x-t)f'(t) \Big|_{t=a}^x + \int_a^x (x-t)f''(t) dt \\ &= f(a)(x-a) + R_1(x). \end{aligned}$$

Now use induction. □

**Theorem 35** (Change of Variable Formula). *Let the map  $x = u(t)$  map the interval  $[c, d]$  into the interval  $[a, b]$  and assume that  $u'(t)$  is integrable on  $[c, d]$ . Then for any continuous function  $f$  on  $[a, b]$*

$$\int_{u(c)}^{u(d)} f(x) dx = \int_c^d f(u(t))u'(t) dt.$$

**Problem 20.** Prove this. *Hint:* Do this in steps

- (a) Explain why both the integrals exist.
- (b) Define  $F$  on  $[a, b]$  by

$$F(x) = \int_a^x f(y) dy$$

and explain why

$$F'(x) = f(x) \quad \text{and} \quad \int_{u(c)}^{u(d)} f(x) dx = F(u(d)) - F(u(c)).$$

- (c) On  $[c, d]$  define

$$G(t) = F(u(t)).$$

By the chain rule

$$G'(t) = F'(u(t))u'(t) = f(u(t))u'(t)$$

and so by Theorem 31

$$\int_c^d f(u(t))u'(t) dt = \int_c^d G'(t) dt = G(d) - G(c).$$

- (d) Put the pieces above together to finish the proof. □

## 3. DEFINITION OF THE LOGARITHM AND EXPONENTIAL FUNCTIONS.

Define a function  $L: (0, \infty) \rightarrow \mathbb{R}$  by

$$L(x) = \int_1^x \frac{dx}{x}.$$

We know this should be the natural logarithm, but we now verify directly from its definition that it has the correct properties.

**Proposition 36.** *The derivative of  $L$  is*

$$L'(x) = \frac{1}{x}$$

*and thus  $L$  is strictly increasing. Therefore  $L$  is one-to-one (that is injective).*

*Proof.* By the Fundamental Theorem of Calculus

$$L'(x) = \frac{1}{x} > 0$$

as  $x > 0$  which implies  $L$  is strictly increasing. □

**Proposition 37.** *Let  $a, b > 0$  then*

$$\int_a^b \frac{dx}{x} = L(b/a).$$

**Problem 21.** Prove this. *Hint:* In the integral  $\int_a^b \frac{dx}{x}$  do the change of variable  $x = at$  to get

$$\int_a^b \frac{dx}{x} = \int_1^{b/a} \frac{dt}{t}. \quad \square$$

**Proposition 38.** *If  $a, b > 0$  then*

$$L(ab) = L(a) + L(b).$$

**Problem 22.** Prove this. *Hint:*

$$L(ab) = \int_1^{ab} \frac{dx}{x} = \int_1^a \frac{dx}{x} + \int_a^{ab} \frac{dx}{x}$$

and use Proposition 37. □

The last Proposition and induction yield:

**Corollary 39.** *If  $a > 0$  and  $n$  is a positive integer*

$$L(a^n) = nL(a). \quad \square$$

**Proposition 40.** *The function  $L: (0, \infty) \rightarrow \mathbb{R}$  is a bijection.*

**Problem 23.** Prove this. *Hint:* Recall the saying that  $L$  is a bijection is just saying that it is one-to-one and onto. We have already seen that  $L$  is injective. To see that it is surjective (that is onto) note that  $L(2) > 0$  and  $L(1/2) < 0$ . Also for a positive integer  $n$

$$L(2^n) = nL(2) \quad \text{and} \quad L(1/2^n) = nL(1/2).$$

If  $y_0$  is any real number we can find (by Archimedes' principle) a positive integer  $n$  such that

$$nL(1/2) < y_0 < nL(2).$$

Also we know that  $L$  is continuous (why?). Now you should be able to show that there is a  $x_0 \in (0, \infty)$  with  $L(x_0) = y_0$ .  $\square$

Because the function  $L: (0, \infty) \rightarrow \mathbb{R}$  is bijective, it has an inverse  $E: \mathbb{R} \rightarrow (0, \infty)$ . As  $L$  is strictly increasing, continuous, and differentiable with  $L'(x) \neq 0$  for all  $x$  theorems from earlier this term imply that  $E$  is strictly increasing, continuous, and differentiable.

**Proposition 41.** *The function  $E$  satisfies  $E(0) = 1$  and*

$$E'(x) = E(x).$$

**Problem 24.** Prove this. *Hint:*  $L(1) = 0$ . And as  $L$  and  $E$  are inverses of each other  $L(E(x)) = x$  for all  $x \in \mathbb{R}$ . Therefore  $\frac{d}{dx} L(E(x)) = 1$ . Use the chain rule and that we know the derivative of  $L$ .  $\square$

**Proposition 42.** *For all real numbers  $x$*

$$E(-x) = \frac{1}{E(x)}.$$

**Problem 25.** Prove this. *Hint:* There are several ways to do this. One is to take the derivative of  $E(x)E(-x)$  and show it is zero. Another is to note that  $L(a) + L(1/a) = L(1) = 0$   $\square$

**Proposition 43.** *For all real numbers  $a, b$*

$$E(a+b) = E(a)E(b).$$

**Problem 26.** Prove this. *Hint:* One way is to deduce this from the property  $L(\alpha\beta) = L(\alpha) + L(\beta)$  of  $L$ . Another is to show that the derivative of the function

$$f(x) = E(x+a)E(-x)$$

is zero and therefore  $f$  is constant.  $\square$

**Proposition 44.** *If  $n$  is any integer, positive or negative, and  $t$  is any real number*

$$E(nt) = E(t)^n$$

*If  $m$  is a positive integer then*

$$E\left(\frac{1}{m}t\right)^m = E(t)$$

*and thus  $E(\frac{1}{m}t)$  is the positive  $m$ -th root of  $E(t)$ .*

**Problem 27.** Prove this. □

In light of Proposition 44 If  $r$  is a rational number, say  $r = n/m$  with  $m, n$  integers and  $m > 0$ , then for a positive number  $a$  we can define

$$a^r = a^{n/m} = (a^n)^{1/m}$$

where  $(a^n)^{1/m}$  is the positive  $m$ -th root of  $a^n$ . We would also like to define  $a^r$  when  $r$  is irrational. Note that when  $r = m/n$  and  $a = E(t)$ , then Proposition 44 shows us that

$$(2) \quad a^r = E(t)^{n/m} = (E(t)^n)^{1/m} = E(nt)^{1/m} = E\left(\frac{1}{m}nt\right) = E(rt).$$

But  $E(rt)$  makes sense for all real numbers  $r$ . We now formalize all this.

**Definition 45.** We now officially define **logarithm** of a positive number  $x$  to be

$$\ln(x) = L(x) = \int_1^x \frac{dt}{t},$$

the number  $e$  to be

$$e = E(1)$$

and for any real number  $x$  we define the power  $e^x$  by

$$e^x = E(x). \quad \square$$

**Definition 46.** Let  $a > 0$ . Then for any real number  $r$  define

$$a^r = e^{r \ln(a)}.$$

(Note if  $a = E(t) = e^t$  then  $\ln(a) = t$  and this becomes  $a^r = e^{r \ln(a)} = e^{rt} = E(rt)$  which agrees with our preliminary definition (2).) □

**Proposition 47.** If  $a > 0$  and  $r = n/m$  is a rational number with  $m > 0$ , then

$$a^r = (a^n)^{1/m}$$

so that our definition agrees with what it should be on the rational numbers. In particular  $a^{1/2}$  is the square root of  $a$ ,  $a^{1/3}$  is the cube root of  $a$  etc.

**Problem 28.** Prove this. □

**Proposition 48.** With these definition the following hold

(a) If  $a > 0$  then for all  $r, s \in \mathbb{R}$

$$a^r a^s = a^{r+s}, \quad \frac{a^r}{a^s} = a^{r-s}.$$

and,

$$(a^r)^s = a^{rs}.$$

(b) If  $r \in \mathbb{R}$  and  $a, b > 0$  then

$$a^r b^r = (ab)^r.$$



(c) If  $r, s \in \mathbb{R}$  and  $a > 0$ , then

$$(a^r)^s = a^{rs}.$$

**Problem 29.** Prove this. □

**Proposition 49.** Let  $r$  be a real number and on define  $f: (0, \infty) \rightarrow (0, \infty)$  by

$$f(x) = x^r.$$

Then  $f$  is differentiable and

$$f'(x) = rx^{r-1}.$$

**Problem 30.** Prove this. *Hint:* We know that  $E(x) = e^x$  is differentiable with derivative  $E'(x) = E(x)$  and that  $\ln(x)$  is differentiable with  $\frac{d}{dx} \ln(x) = 1/x$ . Thus  $f(x) = e^{r \ln(x)} = E(r \ln(x))$  is a composition of differentiable functions. Use the chain rule to derive the formula for  $f'(x)$ . □

**Proposition 50.** Let  $a$  be a positive real number and define  $g: \mathbb{R} \rightarrow (0, \infty)$  by

$$g(x) = a^x.$$

Then  $g$  is differentiable and

$$g'(x) = \ln(a)a^x.$$

**Problem 31.** Prove this. □

There is another way to define  $e^x$  based on the following

**Proposition 51.** For any  $x \in \mathbb{R}$

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x.$$

**Problem 32.** Here is one method of proving this.

- (a) Use Taylor's theorem with the Lagrange form of the remainder to show that for  $|y| \leq 1/2$  that

$$\ln(1 + y) = y + R(y)$$

where

$$|R(y)| \leq 2y^2.$$

- (b) Let  $x \in \mathbb{R}$  and note that if  $|x/n| \leq 1/2$ , we have

$$\ln(1 + x/n) = \frac{x}{n} + R(x/n)$$

and

$$|R(x/n)| \leq \frac{2x^2}{n^2}$$

- (c) Use (b) to show

$$\lim_{n \rightarrow \infty} n \ln(1 + x/n) = x.$$

(d) Now use that

$$\left(1 + \frac{x}{n}\right)^n = e^{n \ln(1+x/n)}$$

to show that

$$\lim_{n \rightarrow \infty} \left(1 + \frac{x}{n}\right)^n = e^x$$

holds. □

There are books that instead of defining the  $\ln(x)$  as  $\int_1^x dt/t$ , and then defining  $e^x$  as the inverse of  $\ln(x)$ , first define  $e^x$  as

$$e^x = \left(1 + \frac{x}{n}\right)^n,$$

show that this behaves like  $e^x$  should and then define  $\ln(x)$  as the inverse of  $e^x$  and finally  $a^r = e^{r \ln(a)}$ . This takes more work than what we have done, but has the advantage that it is possible to define  $e^x$ ,  $\ln(x)$ , and  $a^x$  before defining the derivative and integral.