Mathematics 555 Homework.

Proposition 1. *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be two convergent series and let* $c_1, c_2 \in \mathbf{R}$*. Then the series* $\sum_{k=1}^{\infty}(c_1a_k + c_2b_k)$ *is also convergent and*

$$\sum_{k=1}^{\infty}(c_1a_k + c_2b_k) = c_1\sum_{k=1}^{\infty} a_k + c_2\sum_{k=1}^{\infty} b_k.$$

Problem 1. Prove this. *Hint:* Let $A_n = \sum_{k=1}^{n} a_k$, $B_n = \sum_{k=1}^{n} b_k$, and $S_n = \sum_{k=1}^{n}(c_1a_k + c_2b_k)$ be the partial sums of the series in question. Start by showing $S_n = c_1A_n + c_2B_n$. □

Problem 2. Determine the set of points where the following series converge absolutely and for which values they converge conditionally.

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{(x-3)^n}{n^2 4^n}$,

(b) $\displaystyle\sum_{k=0}^{\infty}(1+\cos(x))^k$,

(c) $\displaystyle\sum_{j=1}^{\infty}\frac{(-1)^j x^{2j+1}}{7^j\sqrt{j}}$

Problem 3. In this problem we show that the function e^x can be expressed as a power series

(a) For each positive integer n and x any real number show

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \frac{e^{\xi}x^{n+1}}{(n+1)!}$$

where ξ is between 0 and x.

(b) With this notation show

$$\lim_{n\to\infty}\frac{e^{\xi}x^{n+1}}{(n+1)!} = 0.$$

Hint: One way to start is by noticing

$$\left|\frac{e^{\xi}x^{n+1}}{(n+1)!}\right| \le \frac{e^{|x|}|x|^{n+1}}{(n+1)!},$$

that the series

$$\sum_{n=0}^{\infty}\frac{e^{|x|}|x|^{n+1}}{(n+1)!}$$

converges (by ratio test), and finally that the terms of a convergent series converge to zero.

(c) Show

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\cdots$$

□

Problem 4. Prove that e is an irrational number. *Hint:* By the last problem

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \cdots$$

Towards a contradiction assume that e is rational, say that

$$e = \frac{p}{q}$$

where p and q are positive integers. Then

$$\frac{p}{q} = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{1}{q!} + \frac{1}{(q+1)!} + \frac{1}{(q+2)!} + \cdots$$

Multiply this by $q!$ to get

$$\begin{aligned} p(q-1)! &= q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!} + \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots \\ &= N + R \end{aligned}$$

where

$$N = q! + q! + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!}$$

and

$$R = \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \cdots$$

Show N is an integer. We now estimate R by comparing it with a geometric series

$$\begin{aligned} R &= \frac{q!}{(q+1)!} + \frac{q!}{(q+2)!} + \frac{q!}{(q+3)!} + \frac{q!}{(q+4)!} + \cdots \\ &= \frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} + \cdots \\ &< \frac{1}{(q+1)} + \frac{1}{(q+1)^2} + \frac{1}{(q+1)^3} + \frac{1}{(q+1)^4} + \frac{1}{(q+1)^5} + \cdots \end{aligned}$$

Sum this last series and use the result to show

$$0 < R < \frac{1}{q} \leq 1.$$

But we also have

$$R = p(q-1)! - N$$

which implies that R is an integer. Explain why this is a contradiction. □

Problem 5. This is to complete the proof of Newton's version of the binomial theorem for real numbers as the exponent. Let α be any real number. Then for nonnegative integers k define

$$\binom{\alpha}{k} = \begin{cases} 1, & k = 0; \\ \dfrac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}, & k \geq 1. \end{cases}$$

When $\alpha = n$ this is the usual binomial coefficient. Define a power series, f_α, by

$$f_\alpha(x) = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

(a) Prove that the radius of convergence of this series is $R = 1$ and therefore f_α is infinitely differentiable on the interval $(-1, 1)$. *Hint:* We proved this in class so you don't have to hand it in.

(b) Prove the ***Pascal identity***

$$\binom{\alpha}{k-1} + \binom{\alpha}{k} = \binom{\alpha+1}{k}.$$

(c) Use the Pascal identity to show

$$(1 + x)f_\alpha(x) = f_{\alpha+1}(x).$$

(d) Prove for $k \geq 1$ that

$$k\binom{\alpha}{k} = \alpha\binom{\alpha-1}{k-1}$$

and use this to show

$$f'_\alpha(x) = \alpha f_{\alpha-1}(x).$$

(e) Use that has just been shown to prove prove

$$\frac{d}{dx}\big((1 + x)^{-\alpha} f_\alpha(x)\big) = 0$$

and use this to show

$$f_\alpha(x) = (1 + x)^\alpha.$$

As was said in class this proof is motivated by the method of integrating factors from which you saw in your class on differential equations. □

The last problem can be summarized as:

Theorem 2 (Newton's Bionomial Theorem)**.** *Let α be any real number then for $|x| < 1$ the power $(1 + x)^\alpha$ has the absolutely convergent power series expansion*

$$(1 + x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k.$$

□

It is not hard to generalize this to powers $(a+b)^\alpha$ where $a > 0$ and $|b| < a$. For then setting $x = b/a$ (which satisfies $|x| < 1$) and therefore

$$\begin{aligned}(a+b)^\alpha &= a^\alpha \left(1 + \frac{b}{a}\right)^\alpha \\ &= a^\alpha \sum_{k=0}^{\infty} \binom{\alpha}{k} \left(\frac{b}{a}\right)^k \\ &= \sum_{k=0}^{\infty} \binom{\alpha}{k} a^{\alpha-k} b^k\end{aligned}$$

which looks more like the usual form of the binomial theorem.

It should be mentioned that there is anther natural way to prove Newton's result. By Taylor's Theorem with Lagrange's form of the remainder we have

$$(1+x)^\alpha = \sum_{k=0}^{n} \binom{\alpha}{k} x^k + \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+\theta x)^{\alpha-n-1}}{(n+1)!} x^{n+1}$$

where $0 < \theta < 1$. Now try to show directly that

$$\lim_{n\to\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n)(1+\theta x)^{\alpha-n-1}}{(n+1)!} x^{n+1} = 0$$

This is much more work than the proof outlined in Problem 5.