Series.

Contents

The material here corresponds to parts of Chaper VII Rosenlicht.

1. Basic definitions and results about series.

We now wish to make sense out of infinite sums

$$\sum_{k=1}^{\infty} = a_1 + a_2 + a_3 + \cdots$$

Definition 1. Let $\langle a_k \rangle_{k=n_0}^{\infty}$ be a sequence of real numbers. The corresponding ***infinite series*** is (or just ***series***) is the sum

$$\sum_{k=k_0}^{\infty} a_k = a_{k_0} + a_{k_0+1} + a_{k_0+2} + \cdots .$$

The n-th ***partial sum*** of the series is

$$A_n = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \cdots + a_{n-1} + a_n = \sum_{k=n_0}^{n} a_k.$$

We say the series ***converges*** and has sum A iff

$$\lim_{n\to\infty} A_n = A.$$

If $\sum_{k=1}^{\infty} a_k$ does not converge, it ***diverges***. □

To make notation easier, when proving results about series we will usually let $n_0 = 0$ or $n_0 = 1$.

Here is a result that follows at once from the facts about limits of sequences.

Theorem 2. *If $\sum_{n=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ both converge, then for any constants c_1 and c_2 the series $\sum_{k=1}^{\infty}(c_1 a a_k + c_2 b_k)$ also converges and*

$$\sum_{k=1}^{\infty}(c_1 a a_k + c_2 b_k) = c_1 \sum_{n=1}^{\infty} a_k + c_2 \sum_{n=1}^{\infty} b_k$$

Proof. Let

$$\begin{aligned}
A_n &= (a_1 + \cdots a_n) \\
B_n &= (b_1 + \cdots b_n) \\
C_n &= \big((c_1a_1 + c_2b_1) + \cdots + (c_1a_n + c_2a_n)\big)
\end{aligned}$$

be the partial sums of the series. We are given that

$$\lim_{n\to\infty} A_n = A, \qquad \lim_{n\to\infty} B_n = B$$

exist and want to show $\lim_{n\to\infty} C_n = c_1A + c_2B$. Note

$$\begin{aligned}
C_n &= \big((c_1a_1 + c_2b_1) + \cdots + (c_1a_n + c_2a_n)\big) \\
&= c_1(a_1 + \cdots a_n) + c_2(b_1 + \cdots b_n) \\
&= c_1A_n + c_2B_n
\end{aligned}$$

and therefore

$$\lim_{n\to\infty} C_n = \lim_{n\to\infty}(c_1A_n + c_2B_n) = c_1A + c_2B$$

as required. □

Before going on we note that for any series $\sum_{k=1}^{\infty} a_k$ with partial sums $A_n = \sum_{k=1}^{n}$ we have the elementary relation

$$A_n = A_{n-1} + a_n,$$

or equivalently

$$a_n = A_n - A_{n-1}.$$

This will come up several times in what follows starting with the following:

Theorem 3. *If the series $\sum_{k=1}^{n} a_k$ converges, then*

$$\lim_{n\to\infty} a_n = 0.$$

Proof. If $A_n = \sum_{k=1}^{n} a_k$ then $\lim_{n\to\infty} A_n = A$ exists as the series converges. But then also $\lim_{n\to\infty} A_{n-1} = A$ and so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}(A_n - A_{n-1}) = A - A = 0.$$

□

Remark 4. Often the previous theorem is used in its contrapositive form: If $\lim_{k\to\infty} a_k \neq 0$, then $\sum_{k=1}^{\infty} a_k$ diverges. From this it is not hard to give lots of examples of series that do not converge. For example none of the following converge

$$\sum_{k=1}^{\infty}(-1)^k, \qquad \sum_{k=1}^{\infty}\sin(k), \qquad \sum_{n=1}^{\infty}\frac{n^2-2}{2n^2+5}.$$ □

Proposition 5. *The series $\sum_{k=1}^{\infty} a_k$ converges if and only if for all $\varepsilon > 0$ there is a N such that*

$$N \le m < n \qquad \Longrightarrow \qquad |a_{m+1} + a_{m+2} \cdots + a_n| < \varepsilon.$$

Problem 1. Prove this. *Hint:* What is the Cauchy condition for the sequence $\langle A_n \rangle_{n=1}^{\infty}$ of partial sums? $\square$

Proposition 6. *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series such that $a_k = b_k$ except for a finite number of values k. Then either they both converge or both diverge. (An informal way so state this is that changing a finite number of terms of a series does not effect whether it converges or diverges.)*

Proof. By the hypothesis there is an n_0 such that such that

$$a_k = b_k \qquad \text{for all} \qquad k \geq n_0.$$

If $n \geq n_0$ then

$$\begin{aligned} B_n &= B_{n_0} + \sum_{k=n_0+1}^{n} b_k \\ &= B_{n_0} + \sum_{k=n_0+1}^{n} a_k && \text{(as } a_k = b_k \text{ when } k \geq n_0) \\ &= B_{n_0} - A_{n_0} + A_{n_0} + \sum_{k=n_0+1}^{n} a_k \\ &= (B_{n_0} - A_{n_0}) + A_n. \end{aligned}$$

Letting $c = B_{n_0} - A_{n_0}$, which is a constant, we have that $B_n = A_n + c$ for $n \geq n_0$. Thus the sequences $\langle A_n \rangle_{n=1}^{\infty}$ and $\langle B_n \rangle_{n=1}^{\infty}$ either both converge or both diverge. $\square$

Lemma 7. *If $|r| \neq 1$ then*

$$a + ar + ar^2 + \cdots ar^n = \sum_{k=0}^{n} ar^k = \frac{a - ar^{n-1}}{1 - r}.$$

Proof. Let $S_n = a + ar + ar^2 + \cdots ar^n$. Then

$$\begin{aligned} (1-r)S_n &= a + ar + ar^2 + \cdots + ar^n - r(a + ar + ar^2 + \cdots + ar^n) \\ &= a + ar + ar^2 + \cdots ar^n - ar - ar^2 - \cdots - ar^n - ar^{n+1} \\ &= a - ar^{n+1}. \end{aligned}$$

As $r \neq 1$ we can divide by $(1 - r)$ to get the desired result. $\square$

Lemma 8. *If $|r| < 1$ then*

$$\lim_{n\to\infty} |r|^n = 0.$$

Proof. Let $\varepsilon > 0$ and set $N = \ln(\varepsilon)/\ln(|r|)$. Then if $n > N$ it is not hard to check $\big||r|^n - 0\big| = |r|^n < \varepsilon$. $\square$

Here one of the most basic examples of series. Many results about series involve comparison to a geometric series.

Theorem 9 (Infinite Geometric Series)**.** *Let a, r be real numbers with $a \neq 0$. Then the series*

$$a + ar + ar^2 + \cdots = \sum_{k=0}^{\infty} ar^k$$

converges if and only if $|r| < 1$ in which case its sum is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}.$$

Proof. If $|r| \geq 1$ then the n-th term ar^n satisfies $|ar^n| \geq |a| > 0$ and so $\lim_{n\to\infty} ar^n \neq 0$ and thus the series diverges.

Now assume $|r| < 1$. We have seem in Lemma 7 that the nth partial sum is

$$S_n = \frac{a - ar^{n+1}}{1-r}.$$

Now by the last lemma,

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a - ar^{n+1}}{1-r} = \frac{a - a \cdot 0}{1-r} = \frac{a}{1-r}$$

as required. □

2. Series with positive terms.

Theorem 10. *Let $\sum_{k=1}^{\infty} a_k$ be a series with $a_k \geq 0$ for all k. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if the sequence, $\langle A_n \rangle_{n=1}^{\infty}$ (with $A_n = a_1 + \cdots a_n$) of partial sums is bounded.*

Proof. If $\sum_{k=1}^{\infty} a_k$ converges, then $\lim_{n\to\infty} A_n = A$ exists by definition. But a convergent sequence is bounded. If $\langle A_n \rangle_{n=1}^{\infty}$ is bounded, then $A_{n+1} = A_n + a_{n+1} \geq A_n$ so the series is monotone increasing. But a bounded monotone sequence is convergent. □

Remark 11. When talking about series, $\sum_{k=1}^{\infty} a_k$, of non-negative terms we will use the following suggestive notation.

$$\sum_{k=1}^{\infty} a_k < \infty \iff \text{The series converges}$$

$$\sum_{k=1}^{\infty} a_k = \infty \iff \text{The series series diverges.}$$

This notation is not appropriate when talking about series with terms of mixed signs. For example the series $\sum_{k=1}^{\infty} (-1)^{k+1}$ has bounded partial sums, but is not convergent. □

3. Tests for the convergence of series with monotone terms.

In general it is easier to understand the convergence of series with monotone decreasing terms. As a first example.

Theorem 12 (Cauchy Condensation Test)**.** *If $\langle a_k\rangle_{k=1}^\infty$ is a sequence of non-negative numbers that are monotone decreasing, then*

$$\sum_{k=1}^\infty a_k < \infty$$

if and only if

$$\sum_{k=0}^\infty 2^k a_{2^k} < \infty.$$

Proof. Let the partial sums of the two series be

$$A_n = \sum_{k=1}^n a_k, \qquad B_n = \sum_{k=0}^n 2^k a_{2^k}.$$

We will show

$$A_{2^{n+1}-1} \le B_n \tag{1}$$

$$B_n \le 2A_{2^n}. \tag{2}$$

If these hold the result is easy. If $\sum_{k=0}^\infty 2^k a_{2^k} < \infty$ then for any positive integer m choose n such that $m \le 2^{n+1}-1$. By (1),

$$A_m \le A_{2^{n+1}-1} \le B_n \le \sum_{k=0}^\infty 2^k a_{2^k} < \infty$$

and therefore the partial sums of $\sum_{k=1}^\infty a_k$ are bounded above and thus $\sum_{k=0}^\infty a_k < \infty$.

Conversely if $\sum_{k=1}^\infty a_k < \infty$ then for any positive integer n we use (2) to get

$$B_n \le 2A_{2^n} \le 2\sum_{k=1}^\infty a_k < \infty$$

which shows the partial sums of $\sum_{k=0}^\infty 2^k a_{2^k}$ are bounded above and thus $\sum_{k=0}^\infty 2^k a_{2^k}$ converges.

We now prove (1). Using that the terms are monotone decreasing,

$$\begin{aligned}
A_{2^{n+1}-1} &= a_1 + \underbrace{(a_2+a_3)}_{2^1 \text{ terms}} + \underbrace{(a_4+\cdots+a_7)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^n}+\cdots+a_{2^{n+1}-1})}_{2^n \text{ terms}} \\
&\le a_1 + \underbrace{(a_2+a_2)}_{2^1 \text{ terms}} + \underbrace{(a_4+\cdots+a_4)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^n}+\cdots+a_{2^n})}_{2^n \text{ terms}} \\
&= a_1 + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots + 2^n a_{2^n} \\
&= B_n.
\end{aligned}$$

The proof (2) is similar

$$\begin{aligned}
A_{2^n} &= a_1 + a_2 + \underbrace{(a_3 + a_4)}_{2^1 \text{ terms}} + \underbrace{(a_5 + \cdots a_8)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^{n-1}+1} + \cdots + a_{2^n})}_{2^{n-1} \text{ terms}} \\
&\geq a_1 + a_2 + \underbrace{(a_4 + a_4)}_{2^1 \text{ terms}} + \underbrace{(a_8 + \cdots a_8)}_{2^2 \text{ terms}} + \cdots + \underbrace{(a_{2^n} + \cdots + a_{2^n})}_{2^{n-1} \text{ terms}} \\
&= a_1 + a_2 + 2^1 a_{2^2} + 2^2 a_{2^3} + \cdots + 2^{n-1} a_{2^n} \\
&= 2^{-1} a_1 + 2^{-1} a_1 + a_2 + 2^1 a_{2^2} + 2^2 a_{2^3} + \cdots + 2^{n-1} a_{2^n} \\
&= 2^{-1} a_1 + 2^{-1} \left(2^0 a_1 + 2^1 a_2 + 2^2 a_{2^2} + 2^3 a_{2^3} + \cdots + 2^n a_{2^n} \right) \\
&= 2^{-1} a_1 + 2^{-1} B_n \\
&\geq \frac{1}{2} B_n.
\end{aligned}$$

Multiplication by 2 completes the proof. □

Theorem 13. *For any real number $p > 0$ the series*

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if and only if $p > 1$.

Proof. We use the Cauchy-Condensation Test, which applies as the terms of the series are decreasing. The given series converges if and only if

$$\sum_{k=1}^{\infty} 2^k \frac{1}{(2^k)^p} = \sum_{k=1}^{\infty} \left(\frac{2}{2^p} \right)^k$$

converges. This is a geometric series with ratio

$$r = \frac{2}{2^p}.$$

Therefore the series converges if and only if $r = 2/2^p < 1$, that is if and only if $p > 1$. □

Anther method of dealing with series with monotone terms is by comparison with an integral. Let us start with an example. Let $f(x)$ be monotone decreasing on the interval $[0, 6]$ and let

$$a_k = f(k) \qquad \text{for} \qquad 1 \leq k \leq 6$$

and

$$A_n = a_1 + \cdots + a_n = f(1) + \cdots + f(n).$$

Then, see Figure 1, we can compare the integral $\int_1^6 f(x)\,dx$ with some of the Riemann sums for the partition $\mathcal{P} = \{1, 2, 3, 4, 5, 6\}$ to get

$$\int_1^6 f(x)\,dx \leq A_5 \leq A_6 \leq f(1) + \int_1^6 f(x)\,dx.$$

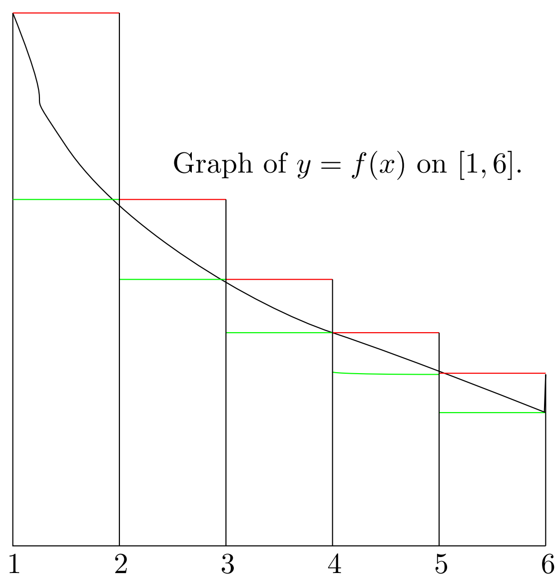

FIGURE 1. The area under the tall (with red tops) rectangles is $A_5 = f(1) + f(2) + f(3) + f(4) + f(5)$. The area under the short (with green tops) rectangles is $A_6 - f(1) = f(2) + f(3) + f(4) + f(5) + f(6)$. The area of the integral is clearly in between these two areas and therefore

$$A_6 - f(1) \leq \int_1^6 f(x)\,dx \leq A_5.$$

This can be rearranged to give

$$\int_1^6 f(x)\,dx \leq A_5 \leq A_6 \leq f(1) + \int_1^6 f(x)\,dx = a_1 + \int_1^6 f(x)\,dx$$

which is a bit more aesthetic.

We could, and since this is a mathematics class, should be a bit more formal. Note that on any interval $[k, k+1]$ we have, because f is decreasing, that

$$f(k) \geq f(x) \geq f(k+1).$$

Then integration over $[k, k+1]$ and using that $\int_k^{k+1} f(k)\,dx = f(k)$ and $\int_k^{k+1} f(k+1)\,dx = f(k+1)$

$$f(k) \geq \int_k^{k+1} f(x)\,dx \geq f(k+1).$$

This can be summed it two ways to get

$$\int_1^6 f(x)\,dx = \sum_{k=1}^{5} \int_k^{k+1} f(x)\,dx \leq \sum_{k=1}^{5} f(k) = A_5$$

and

$$A_6 - a_1 = \sum_{k=2}^{6} f(k) \le \sum_{k=1}^{5} \int_k^{k+1} f(x)\,dx = \int_1^6 f(x)\,dx.$$

Of course there is nothing special about $n = 6$ in this argument.

Proposition 14. *Let $f\colon [1,\infty) \to [0,\infty)$ be a monotone decreasing non-negative function. Let $a_k = f(k)$ and let*

$$A_n = \sum_{k=1}^{n} a_k$$

be the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Then

$$\int_1^n f(x)\,dx \le A_n \le f(1) + \int_1^n f(x)\,dx.$$

Problem 2. Use a variation of the argument given for $n = 6$ to prove this. □

Theorem 15 (The Integral Test)**.** *Let $f\colon [1,\infty) \to [0,\infty)$ be a monotone decreasing non-negative function. Let $a_k = f(k)$ and let*

$$A_n = \sum_{k=1}^{n} a_k$$

be the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$. Then

$$\sum_{k=1}^{\infty} a_k < \infty \qquad \Longleftrightarrow \qquad \lim_{n\to\infty} \int_1^n f(x)\,dx \quad \text{exists and is finite.}$$

(Note that $\left\langle \int_1^n f(x)\,dx \right\rangle_{n=1}^{\infty}$ is a monotone increasing sequence, thus the limit exists, but might be $+\infty$.)

Problem 3. Prove this. □

Problem 4. Use the Integral Test to give anther proof of Theorem 13. □

Problem 5. Use the Integral Test to show

$$\sum_{k=2}^{\infty} \frac{1}{n(\ln(n))^p}$$

converges if and only if $p > 1$. □

4. Comparison tests.

Proposition 16. *Let Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be two series of non-negative terms. Assume there is a constant $C > 0$ such that*

$$a_k \le C b_k$$

for all k. Then

(a) If $\sum_{k=1}^{\infty} b_k$ *converges, so does* $\sum_{k=1}^{\infty} a_k$.

(b) If $\sum_{k=1}^{\infty} a_k$ *diverges, so does* $\sum_{k=1}^{\infty} b_k$.

Problem 6. Prove this. *Hint:* Consider partial sums. □

Theorem 17 (Limit Comparison Test)**.** *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be two series of positive terms. Assume that*

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

exists. Then

(a) $\sum_{k=1}^{\infty} b_k < \infty$ *implies* $\sum_{k=1}^{\infty} a_k < \infty$

(b) If $L \neq 0$ *and* $\sum_{k=1}^{\infty} a_k = \infty$, *then* $\sum_{k=1}^{\infty} b_k = \infty$.

Often the following special case is enough.

Corollary 18. *Let* $\sum_{k=1}^{\infty} a_k$ *and* $\sum_{k=1}^{\infty} b_k$ *be two series of positive terms. Assume that*

$$L = \lim_{k\to\infty} \frac{a_k}{b_k}$$

exists and $L \neq 0$. *Then* $\sum_{k=1}^{\infty} a_k$ *converges if and only if* $\sum_{k=1}^{\infty} b_k$ *converges.* □

Problem 7. Prove Theorem 17. *Hint:* Recall that a convergent sequence is bounded. Thus $\langle a_k/b_k\rangle_{k=1}^{\infty}$ is bounded and therefore there is a constant C such that $a_k/b_k \leq C$. Thus Proposition 16 applies.

Here some applications of these results.

Example 19. Does the series $\sum_{k=1}^{\infty} \frac{k^3+2k^2+7}{3k^5+2}$ converge? Let this series be $\sum_{k=1}^{\infty} a_k$ and let $\sum_{k=1}^{\infty} b_n$ be the p-series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Then it is not hard to check that

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{1}{3}.$$

Therefore, by Corollary 18, $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges. But $\sum_{k=1}^{\infty} b_k$ is a p series with $p = 2 > 1$ and so both series converge. □

Example 20. Does the series $\sum_{k=1}^{\infty} = \sum_{k=1}^{\infty} \left(\sqrt[3]{n+5} - \sqrt[3]{n-2}\right)$ converge? Let $f(x) = \sqrt[3]{x} = x^{1/3}$. Then for $n > 2$ by the mean value theorem there is a ξ_n between -2 and 5 such that

$$a_n = f(n+5) - f(n-2) = f'(n+\xi_n)((n+5)-(n-2)) = \frac{1}{3}(n+\xi_n)^{-2/3}7.$$

Therefore if $\sum_{k=1}^{\infty} b_k$ is the divergent p-series $\sum_{k=1}^{\infty} 1/n^{2/3}$ we have

$$\lim_{k\to\infty} \frac{a_k}{b_k} = \frac{7}{3}.$$

So $\sum_{k=1}^{\infty} a_k$ diverges by limit comparison to $\sum_{k=1}^{\infty} b_k$.

Problem 8*.* For practice in these ideas do Problems 10 and 11 on Page 161 of the text. *Hint:* For Problem 11 it might help to notice that

$$\frac{1}{n} - \frac{1}{n+x} = \frac{x}{n(n+x)} \quad \text{and} \quad \lim_{n\to\infty} \frac{1/n^2}{1/(n(n+x))} = 1. \qquad \square$$

5. The root and ratio tests

These are basically just limit comparisons with a geometric series. To get started here is a version of the comparison were we only worry about the comparison for large values.

Lemma 21. *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series of positive terms. Assume there is an N such that*

$$a_k \le b_k \qquad \text{for all} \qquad k > N$$

and that $\sum_{k=1}^{\infty} b_k < \infty$. Then $\sum_{k=1}^{\infty} a_k < \infty$.

Proof. Let A_n and B_n be the partial sums of these series. Let

$$C_1 = \max\{A_n : 1 \le n \le N\}.$$

If $n > N$ then

$$\begin{aligned}
A_n &= (a_1 + \cdots a_N) + (a_{N+1} + \cdots + a_n) \\
&\le (a_1 + \cdots a_N) + (b_{N+1} + \cdots + b_n) \\
&= (a_1 + \cdots a_N) - (b_1 + \cdots + b_N) + (b_1 + \cdots + b_N + b_{N+1} + \cdots + b_n) \\
&= A_N - B_N + B_n \\
&\le A_N - B_N + \sum_{k=1}^{\infty} b_k < \infty.
\end{aligned}$$

Therefore if

$$C = \max\left\{ C_1, A_N - B_N + \sum_{k=1}^{\infty} b_k \right\}$$

we have

$$A_n \le C$$

for all n. Thus the partial sums of $\sum_{k=1}^{\infty} a_k$ are bounded which implies that it is convergent. $\square$

The following is a dressed up version of doing a comparison with a geometric series.

Theorem 22 (Root Test)**.** *Let $\sum_{k=1}^{\infty} a_k$ be a series of positive terms and assume the limit*

$$\rho := \lim_{k\to\infty} (a_k)^{1/k}.$$

exists.

(a) *If $\rho < 1$ then the series converges.*
(b) *If $\rho > 1$ then the series diverges.*

Problem 9. Prove this. *Hint:* For (a) let r be any number such that $\rho < r < 1$. Then $\rho = \lim_{k\to\infty}(a_k)^{1/k} < r$ implies there is a N such that

$$k > N \qquad \Longrightarrow \qquad (a_k)^{1/k} < r.$$

Then

$$a_k < r^k \qquad \text{for all} \qquad k > N.$$

Now consider Lemma 21 and Theorem 9.

For (b) show that if $\rho > 1$ then $\lim_{k\to\infty} a_k \neq 0$. □

Here is anther dressed up version of comparison with a geometric series.

Theorem 23 (Ratio Test)**.** *Let $\sum_{k=1}^{\infty} a_k$ be a series of positive terms assume the limit*

$$\rho := \lim_{k\to\infty} \frac{a_{k+1}}{a_k}$$

exists.

(a) *If $\rho < 1$, then the series converges.*
(b) *If $\rho > 1$, then the series diverges.*

Problem 10. Prove this. *Hint:* For (a) let r be a number such that $\rho < r < 1$. Then, by the definition of lim, there is a N such that

$$k > N \qquad \Longrightarrow \qquad \frac{a_{k+1}}{a_k} < r.$$

Thus for $k > N$ we have

$$a_k = a_{N+1}\frac{a_{N+2}}{a_{N+1}}\frac{a_{N+3}}{a_{N+2}}\cdots\frac{a_{k-1}}{a_{k-2}}\frac{a_k}{a_{k-1}} = (a_{N+1})\prod_{j=N+1}^{k-1}\frac{a_{j+1}}{a_j} < a_{N+1}r^{k-N-1}.$$

The series

$$\sum_{k=1}^{\infty}(a_{N+1})r^{k-N-1} = \sum_{k=1}^{\infty}\left(a_{N+1}r^{-N-1}\right)r^k = \sum_{k=1}^{\infty} Cr^k$$

(where $C = \left(a_{N+1}r^{-N-1}\right)$) is a convergent geometric series. You should now be able to do a comparison by use of Lemma 21.

For (b) show $\rho > 1$ implies $\lim_{k\to\infty} a_k \neq 0$. □

The following shows that if the ratio test works, then the root test will also work.

Proposition 24. *Let* $\langle a_n \rangle_{n=1}^{\infty}$ *be a sequence of positive real numbers such that*
$$\lim_{n\to\infty} \frac{a_n + 1}{a_n} = r$$
exists. Then also
$$\lim_{n\to\infty} a_n^{1/n} = r.$$

Problem 11. Prove this by filling in the details of the following outline of a proof. Let $\varepsilon > 0$. We start by using the same idea as the proof of 23. We choose an N such that
$$n \geq N \quad \text{implies} \quad \left| \frac{a_{n+1}}{a_n} - r \right| < \frac{\varepsilon}{2},$$
which implies
$$r - \frac{\varepsilon}{2} < \frac{a_{n+1}}{a_n} < r + \frac{\varepsilon}{2}.$$
Show for $n > N$
$$a_n = a_N \left(\frac{a_{N+1}}{a_N} \right) \left(\frac{a_{N+2}}{a_{N+1}} \right) \left(\frac{a_{N+3}}{a_{N+2}} \right) \cdots \left(\frac{a_{n-1}}{a_{n-2}} \right) \left(\frac{a_n}{a_{n-1}} \right)$$
and therefore
$$a_N \left(r - \frac{\varepsilon}{2} \right)^{n-N} < a_n < a_N \left(r + \frac{\varepsilon}{2} \right)^{n-N}$$
Taking n-th roots
$$a_N^{1/n} \left(r - \frac{\varepsilon}{2} \right)^{1-N/n} < a^{1/n} < a_N^{1/n} \left(r + \frac{\varepsilon}{2} \right)^{1-N/n}.$$
But
$$\lim_{n\to\infty} a_N^{1/n} \left(r - \frac{\varepsilon}{2} \right)^{1-N/n} = a_N^0 \left(r - \frac{\varepsilon}{2} \right)^{1-0} = \left(r - \frac{\varepsilon}{2} \right).$$
This implies there is $N_1 > N$ such that
$$n \geq N_1 \quad \text{implies} \quad \left| a_N^{1/n} \left(r - \frac{\varepsilon}{2} \right)^{1-N/n} - \left(r - \frac{\varepsilon}{2} \right) \right|$$
which in turn implies
$$r - \varepsilon < a_N^{1/n} \left(r - \frac{\varepsilon}{2} \right)^{1-N/n}.$$
Do a similar argument to show there is a $N_2 > N$ such that
$$n \geq N_2 \quad \text{implies} \quad a_N^{1/n} \left(r + \frac{\varepsilon}{2} \right)^{1-N/n} < r + \varepsilon.$$
Set $N_3 = \max\{N_1, N_2\}$ and put the inequalities above together to get
$$n \geq N_3 \quad \text{implies} \quad \left| a_n^{1/n} - r \right| < \varepsilon$$
which finishes the proof. $\square$

Problem 12. Here are a couple of applications of Proposition 24.

(a) For n a positive integer let
$$a_n = \frac{n!}{n^n}.$$
Show
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1}{e}.$$
Use this and Proposition 24 to show
$$\lim_{n\to\infty} \frac{(n!)^{1/n}}{n} = \frac{1}{e}.$$

(b) Let
$$b_n = \binom{2n}{n} = \frac{(2n)!}{n!\,n!}.$$
Show
$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 4$$
and use this to show
$$\lim_{n\to\infty} \binom{2n}{n}^{1/n} = 4.$$
□

6. Absolutely and conditional convergent series.

Definition 25. The series $\sum_{k=1}^{\infty} a_k$ is ***absolutely convergent*** iff the series of absolute values $\sum_{k=1}^{\infty} |a_k|$ is convergent. □

Theorem 26. *If $\sum_{k=1}^{\infty} a_k$ is absolutely convergent, then it is convergent and*
$$\left| \sum_{k=1}^{\infty} a_k \right| \le \sum_{k=1}^{\infty} |a_k|.$$

Problem 13. Prove this. *Hint:* Proposition 5 and the triangle inequality applied to partial sums. □

This, together with Proposition 16 implies

Proposition 27. *Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with $|a_k| \le Cb_k$ for some positive constant C. Assume $\sum_{k=1}^{\infty} b_k$ converges. Then $\sum_{k=1}^{\infty} a_k$ converges absolutely.* □

Example 28. The last proposition implies all the following
$$\sum_{k=1}^{\infty} \frac{\cos(k)}{k^2}, \qquad \sum_{k=1}^{\infty} \frac{(-1)^k}{n2^n}, \qquad \sum_{k=1}^{\infty} \frac{3+(-1)^k}{(k+1)\ln^2(k+1)}.$$
converge absolutely. □

Definition 29. The series $\sum_{k=1}^{\infty} a_k$ is ***conditional convergent*** iff $\sum_{k=1}^{\infty} a_k$ converges, but $\sum_{k=1}^{\infty} |a_k| = \infty$. □

The following gives one of the main methods of producing conditional convergent series.

Theorem 30. *Let $\langle a_k \rangle_{k=1}^{\infty}$ be a sequence of real numbers with*
(a) $a_k \geq a_{k+1}$ *(that is it is monotone decreasing),*
(b) $\lim_{k\to\infty} a_k = 0$.

Then

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges. If $A = \sum_{k=1}^{\infty} (-1)^{k+1} a_k$ *is the sum and* $A_n = \sum_{k=1}^{n} a_k$ *is the* n*-th partial sum then*

$$|A - A_n| \leq a_{n+1}.$$

That is the error at stopping at the n*-th term is at most the* $(n+1)$*-st term.*

Problem 14. Prove this. *Hint:* Note

$$A_3 = A_1 - a_2 + a_3 = A_1 - (a_2 - a_3) \leq A_1$$

as $a_2 \geq a_3$. Likewise

$$A_5 = A_3 - a_4 + a_5 = A_3 - (a_4 - a_5) \leq A_3$$

as $a_4 \geq a_5$. In general

$$A_{2m+3} = A_{2m+1} - (a_{2m} - a_{2m+1}) \leq A_{2m+1}$$

Give an analogous argument to show

$$A_{2m+2} = A_{2m} + (a_{2m+1} - a_{2m+2}) \geq A_{2m}.$$

Now use this to show that if $\ell \geq n$ then for n odd

$$A_{n+1} \leq A_\ell \leq A_n$$

and for n even

$$A_n \leq A_\ell \leq A_{n+1}.$$

Therefore if $\ell \geq n$ the partial sum A_ℓ is between A_n and A_{n+1}. Also show $|A_{n+1} - A_n| = a_{n+1}$. It should not be hard to finish from here. □

Problem 15. Show that if $0 < p \leq 1$ that the series

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$$

is conditional convergent. □

Therefore when $0 < p \leq 1$ (which implies $\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges) the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$ is conditionally convergent.

7. Power series.

Theorem 31. *Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers and let $f(x)$ be defined on* $\mathbf{R}$ *by*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for all x where this converges. If the series converges for $x = x_0$, then it converges absolutely for all x with $|x| < |x_0|$.

Problem 16*.* Prove this. *Hint:* As

$$f(x_0) = \sum_{k=0}^{\infty} a_k (x_0)^k$$

converges we have $\lim_{k\to\infty} a_k (x_0)^k = 0$ by Theorem 3. This implies that $\langle a_k (x_0)^k \rangle_{k=0}^{\infty}$ is bounded. So there is a constant C with

$$|a_k (x_0)^k| = |a_k||x_0|^k \le C.$$

Then for $|x| < |x_0|$ we have

$$|a_k x^k| = |a_k||x|^k = |a_k||x_0|^k \left(\frac{|x|}{|x_0|}\right)^k \le C \left(\frac{|x|}{|x_0|}\right)^k = C r^k$$

where

$$r = \frac{|x|}{|x_0|} < 1. \qquad \square$$

Lemma 32. *Let $f(x)$ be as in the last theorem. If the series for $f(x)$ converges at $x = x_0$, then the series*

$$f^*(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

*converges absolutely for all x with $|x| < |x_0|$. We call f^* the **formal derivative** of f as it is what the derivative would be if we knew that we could take it term at a time. (Shortly we will show that this the actual derivative.)*

Problem 17*.* Prove this. *Hint:* With notation as in Problem 16 show

$$|k a_k x^{k-1}| \le k C r^{k-1}$$

and then show $\sum_{k=1}^{\infty} k C r^{k-1}$ converges by either the root or ratio test. $\square$

Corollary 33. *With the same hypothesis as in the last lemma for $|x| < |x_0|$ the series*

$$f^{**}(x) = \sum_{k=2}^{\infty} k(k-1) a_k x^{k-2}$$

*converges absolutely. (This is the **formal second derivative**.)*

Proof. As $|x| < |x_0|$ there is a number r_0 such that $|x| < r_0 < |x_0|$. By the lemma the series $f^*(r_0)$ converges absolutely. But (with what I hope is not confusing notation) $(f^*)^*(x) = f^{**}(x)$ so this corollary follows by applying Lemma 32 to f^* (with r_0 replacing x_0). $\square$

Lemma 34. *Let k be a positive integer and x, x_1, r_0 real numbers with $|x|, |x_0| < r_0$. Then*

$$\left| \frac{x^k - x_1^k}{x - x_1} - kx_1^{k-1} \right| \le \frac{k(k-1)}{2} {r_0}^{k-2} |x - x_0|.$$

Problem 18. Prove this. *Hint:* This is yet anther opportunity to use Taylor's theorem. Let $p(x)$ be any two times differentiable function. By Taylor's theorem

$$p(x) = p(x_1) + p'(x_1)(x - x_1) + \frac{p''(\xi)}{2}(x - x_1)^2$$

where ξ is between x and x_1. This can be rearranged as

$$\frac{p(x) - p(x_1)}{x - x_1} - p'(x_1) = \frac{p''(\xi)}{2}(x - x_1)$$

and so

$$\left| \frac{p(x) - p(x_1)}{x - x_1} - p'(x_1) \right| = \frac{|p''(\xi)|}{2}|x - x_1|.$$

Now consider the special case where $p(x) = x^k$. Then $|p''(\xi)| = k(k-1)|\xi|^{k-2} < k(k-1)r_0^{k-2}$ as ξ is between x and x_1 and $|x|, |x_1| < r_0$. $\square$

Theorem 35. *Let $a_0, a_1, a_2, \ldots$ be a sequence of numbers and let $f(x)$ be defined on $\mathbf{R}$ by*

$$f(x) = \sum_{k=0}^{\infty} a_k x^k$$

for all x where this converges. If the series converges for $x = x_0$, then the function $f(x)$ exists and is differentiable for all x with $|x| < |x_0|$ and the derivative is given by the formal derivative

$$f'(x) = f^*(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}.$$

Problem 19. Prove this. *Hint:* That $f(x)$ exists for $|x| < |x_0|$ follows from Theorem 31. We need so show that if $|x_1| < |x_0|$ that f is differentiable at x_1 and the derivative is $f^*(x_1)$. Choose a number r_0 such that $|x_1| < r_0 < |x_0|$. Let x be such that $|x| < r_0$. Explain why the following hold.

(a) The series for the following all converge absolutely.

$$f(x), \quad f(x_1), \quad f^*(x_1), \quad f^{**}(r_0).$$

(b) We have

$$\frac{f(x) - f(x_1)}{x - x_1} - f^*(x_1) = \sum_{k=1}^{\infty} a_k \left(\frac{x^k - x_1^k}{x - x_1} - kx_1^{k-1} \right)$$

(c) The inequality

$$\left|\frac{f(x)-f(x_1)}{x-x_1}-f^*(x_1)\right|\le C|x-x_1|$$

holds, where

$$C=\frac{1}{2}\sum_{k=2}^{\infty}k(k-1)|a_k|r_0^{k-1}<\infty$$

holds. (Part of the problem is explaining why $C<\infty$. The hint here is that the series for $f^{**}(r_0)$ converges absolutely.)

(d) To finish show

$$f'(x_1)=\lim_{x\to x_1}\frac{f(x)-f(x_1)}{x-x_1}=f^*(x_1).$$

□

Now that we have differentiated we wish to integrate. Note that by Theorem 35 if the series $f(x)=\sum_{k=0}^{\infty}a_kx^k$ converges for $x=x_0$, then it is differentiable on the interval $(-|x_0|,||x_0|)$ and therefore also continuous on this interval. Thus if $|x|<|x_0|$ this implies $\int_0^x f(t)\,dt$ is the integral of a continuous function and thus it exists.

Theorem 36. *Let $a_0,a_1,a_2,\dots$ be a sequence of numbers and let $f(x)$ be defined on* $\mathbf{R}$ *by*

$$f(x)=\sum_{k=0}^{\infty}a_kx^k$$

for all x where this converges. If the series converges for $x=x_0$, then for any x with $|x|<|x_0|$

$$\int_0^x f(t)\,dt=\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1}=\sum_{k=1}^{\infty}\frac{a_{k-1}}{k}x^k.$$

That is we can integrate the series for $f(x)$ term at a time.

Problem 20. Prove this. *Hint:* Let $F(x)$ be defined to be the ***formal integral*** of $f(x)$. That is

$$F(x)=\sum_{k=0}^{\infty}\frac{a_k}{k+1}x^{k+1}.$$

Choose r_0 with $|x|<r_0<|x_0|$. Then as the series for $f(x)$ is convergent, its terms are bounded. That is there is a constant C such that

$$|a_kx_0^k|\le C.$$

Then

$$\left|\frac{a_k}{k+1}r_0^{k+1}\right|=\frac{r_0|a_kx_0^k|}{k+1}\left|\frac{r_0}{x_0}\right|^k\le\frac{r_0C}{k+1}\left|\frac{r_0}{x_0}\right|^k=\frac{C_1}{k+1}r^k\le C_1r^k$$

where

$$C_1 = r_0 C \qquad \text{and} \qquad r = \left|\frac{r_0}{x_0}\right| < 1.$$

Now

(a) Explain why the series for $F(r_0)$ converges absolutely. *Hint:* Compare the the geometric series $\sum_{k=0}^{\infty} C_1 r^k$.
(b) Explain why $F(x)$ is differentiable on the interval $(-r_0, r_0)$. *Hint:* Theorem 35 with x_0 replaced by r_0.
(c) The derivative of $F(x)$ on $(-r_0, r_0)$ is $f(x)$ *Hint:* Theorem 35 again.
(d) Finish the proof. *Hint:* Fundamental Theorem of Calculus. □

Now that we know that we can integrate and differentiate power series we can find new series form old ones.

Example 37. Find the series for $(1+x)^{-2}$ on the integral $(-1, 1)$. We know

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 - \cdots$$

This can be differentiated term at a time to get

$$-(1+x)^{-2} = 0 - 1 + 2x - 3x^2 + 4x^3 - 5x^4 + 6x^5 - \cdots$$

so that

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + 5x^4 - 6x^5 - \cdots = \sum_{k=0}^{\infty} (-1)^k (k+1) x^k. \qquad \square$$

Similar examples can be done by integrating term at a time. Here are some for you to try.

Problem 21. (a) Find a series for $\ln(1+x)$ valid on $(-1, 1)$. *Hint:*

$$\ln(1+x) = \int_0^x \frac{dt}{1+t}$$

and you know how to expand $1/(1+t)$ in a series.
(b) For any positive integer n find the series for $(1+x)^{-n}$ valid on $(-1, 1)$.
(c) On $(-1, 1)$ we have the convergent geometric series:

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 - x^{10} + \cdots$$

Use this to find a power series for $\arctan(x)$ valid on $(-1, 1)$. □