Trigonometric functions.

We can now give definitions of the trigonometric functions. It is enough to define sin and cos as all the others can be defined in terms of these two.

Theorem 1. *The two series*

$$\mathsf{c}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!} + \cdots$$
$$\mathsf{s}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \cdots$$

converge absolutely for all $x \in \mathbf{R}$ and therefore these series are absolutely convergent and differentiable for all $x \in \mathbf{R}$. The derivatives satisfy

$$\mathsf{c}'(x) = -\mathsf{s}(x), \qquad \mathsf{s}'(x) = \mathsf{c}(x).$$

The values at $x = 0$ are

$$\mathsf{c}(0) = 1, \qquad \mathsf{s}(0) = 0.$$

Also

$$\mathsf{c}'' = -\mathsf{c}, \qquad \mathsf{s}'' = -\mathsf{s}.$$

Problem 1. Prove this. □

Proposition 2. *These functions satisfy*

$$\mathsf{c}(x)^2 + \mathsf{s}(x)^2 = 1.$$

Problem 2. Prove this. *Hint:* Show that $\mathsf{c}(x)^2 + \mathsf{s}(x)^2$ is constant by taking its derivative. Note that showing it is constant does not finish the problem, you still have to show the constant is 1. □

Lemma 3. *If g is two times differentiable on $\mathbf{R}$ and*

$$g'' = -g, \qquad g(0) = 0, \quad g'(0) = 0$$

then $g(x) = 0$ for all x.

Problem 3. Prove this. *Hint:* Let $E = g^2 + (g')^2$ and show $E' = 0$. □

Theorem 4. *If f is twice differentiable on $\mathbf{R}$ and*

$$f'' = -f$$

then f is a linear combination of c and s. In particular

$$f(x) = f(0)\mathsf{c}(x) + f'(0)\mathsf{s}(x).$$

Problem 4. Prove this. *Hint:* Let $g(x) = f(x) - f(0)\mathsf{c}(x) - f'(0)\mathsf{s}(x)$ and use Lemma 3. □

Theorem 5. *The function c and s satisfy*

$$\mathsf{c}(x+a) = \mathsf{c}(a)\mathsf{c}(x) - \mathsf{s}(a)\mathsf{s}(x)$$
$$\mathsf{s}(x+a) = \mathsf{s}(a)\mathsf{c}(x) + \mathsf{c}(a)\mathsf{s}(x).$$

Problem 5. Prove this. *Hint:* For the first one let $f(x) = \mathsf{c}(x+a)$. Then $f''(x) = -f(x)$. Thus, by Theorem 4,

$$f(x) = f(0)\mathsf{c}(x) + f'(0)\mathsf{s}(x).$$

□

Lemma 6. *If for* $0 < x < 6$ *the inequality*

$$\mathsf{s}(x) \geq x - \frac{x^3}{6}$$

holds.

Proof. For any x we have

$$\begin{aligned}\mathsf{s}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!} + \frac{x^{13}}{13!} - \frac{x^{15}}{15!} + \cdots \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!}\left(1 - \frac{x^2}{6\cdot 7}\right) + \frac{x^9}{9!}\left(1 - \frac{x^2}{10\cdot 11}\right) + \frac{x^{13}}{13!}\left(1 - \frac{x^2}{14\cdot 15}\right) + \cdots\end{aligned}$$

If $0 < x < 6$ then $x^2 < 6\cdot 7 < 10\cdot 11 < 14\cdot 15$. Therefore all the terms

$$\frac{x^5}{5!}\left(1 - \frac{x^2}{6\cdot 7}\right), \quad \frac{x^9}{9!}\left(1 - \frac{x^2}{10\cdot 11}\right), \quad \frac{x^{13}}{13!}\left(1 - \frac{x^2}{14\cdot 15}\right), \ldots$$

are positive and the result follows. □

Lemma 7. *If* $0 < x < 7$ *the inequality*

$$\mathsf{c}(x) < 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

holds.

Problem 6. Prove this. □

Theorem 8. *The function* $\mathsf{c}(x)$ *has a unique zero in the interval* $[0,2]$*. We denote this zero by* $\pi/2$*. This is our official definition of the number* π*.*

Proof. On the interval $(0,2)$ we have, by Lemma 6 that

$$\mathsf{s}(x) \geq x - \frac{x^3}{6} = x\left(1 - \frac{x^2}{6}\right) > 0.$$

Therefore when $0 < x < 2$

$$\mathsf{c}'(x) = -\mathsf{s}(x) < 0.$$

This shows that $\mathsf{c}(x)$ is strictly decreasing on $[0,2]$. Thus $\mathsf{c}(x)$ can have at most one zero on $[0,2]$. But

$$\mathsf{c}(0) = 1$$

and by Lemma 7

$$\mathsf{c}(2) < 1 - \frac{2^2}{2} + \frac{2^4}{24} = \frac{-1}{3} < 0$$

and therefore $\mathsf{c}(x)$ has at least one root in $[0,2]$ by the Intermediate Value Theorem. □

Proposition 9. *The following hold*

$$\begin{aligned} \mathsf{c}(\pi/2) &= 0 & \qquad \mathsf{s}(\pi/2) &= 1 \\ \mathsf{c}(\pi) &= -1 & \mathsf{s}(\pi) &= 0 \\ \mathsf{c}(2\pi) &= 1 & \mathsf{s}(2\pi) &= 0 \end{aligned}$$

Problem 7. Prove this. *Hint:* That $\mathsf{c}(\pi/2) = 0$ is the definition of π. Then $\mathsf{c}(\pi/2)^2 + \mathsf{s}(\pi/2)^2 = 1$ implies $\mathsf{s}(\pi/2) = \pm 1$. Use Lemma 6 to rule out $\mathsf{s}(\pi/2) = -1$. The rest should now follow from Theorem 5. □

Theorem 10. *The following hold.*

$$\begin{aligned} \mathsf{c}(x+\pi/2) &= -\mathsf{s}(x) & \qquad \mathsf{s}(x+\pi/2) &= \mathsf{c}(x) \\ \mathsf{c}(x+\pi) &= -\mathsf{c}(x) & \mathsf{s}(x+\pi) &= -\mathsf{s}(x) \\ \mathsf{c}(x+2\pi) &= \mathsf{c}(x) & \mathsf{s}(x+2\pi) &= \mathsf{s}(x) \end{aligned}$$

Problem 8. Prove this. □

Definition 11. Our official definition of cos and sin is

$$\cos(x) = \mathsf{c}(x), \qquad \sin(x) = \mathsf{s}(x)$$

where c and s are as in Theorem 1. Then $\tan(x)$ is defined by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

at all points where $\cos(x) \neq 0$. The other trigonometric functions (cot, sec, and csc) can also be defined in terms of sin and cos by the usual formulas. □

Proposition 12. *The derivative of* $\tan(x)$ *is given by*

$$\frac{d}{dx}\tan(x) = 1 + \tan^2(x).$$

Also tan *is periodic with period* π*:*

$$\tan(x+\pi) = \tan(x).$$

Problem 9. Prove this. □

Note that $\cos(x) > 0$ on the interval $(-\pi/2, \pi/2)$ and therefore $\tan(x)$ is defined on this interval. Also the derivative $\tan'(x) = 1 + \tan^2(x) \geq 1$ is positive and therefore $\tan(x)$ is strictly increasing on this interval.

Lemma 13. *The one sides limits*

$$\lim_{x\uparrow\pi/2} \tan(x) = +\infty, \qquad \lim_{x\downarrow-\pi/2} \tan(x) = -\infty$$

hold. Therefore the restriction of tan *to* $(-\pi/2, \pi/2)$ *is a bijection (i.e. one-to-one onto function)* $(-\pi/2, \pi/2) \to \mathbf{R}$. *Therefore the restriction of* tan *to* $(-pi/2, \pi/2)$ *has an inverse* $\arctan\colon \mathbf{R} \to (-\pi/2, \pi/2)$.

Problem 10. Prove this. □

Problem 11. Use that $\tan'(x) = 1 + \tan^2(x)$ to show

$$\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2} = \sum_{k=0}^{\infty}(-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + x^8 - \cdots$$

and integrate this to show

$$\arctan(x) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{2k+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^5}{7} + \cdots$$

and that this series has radius of convergence $R = 1$. □

Remark 14. To compute π we can use the series for the arctan For this to be efficient we wish to use values of x that are close to zero. To get a reasonably rapidly convergent series note if

$$\alpha = \arctan(1/2), \qquad \beta = \arctan(1/3)$$

then using the addition angle for the tangent we havd

$$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)} = \frac{1/2+1/3}{1-(1/2)(1/3)} = 1.$$

Therefore

$$\alpha + \beta = \frac{\pi}{4}$$

which implies

$$\pi = 4\arctan\left(\frac{1}{2}\right) + 4\arctan\left(\frac{1}{3}\right) = 4\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(\frac{1}{2^{2k+1}} + \frac{1}{3^{2k+1}}\right).$$

Stopping this series at $k = 13$ gives the value of π to 10 decimal places.

In 1796 John Machin showed that[1]

$$\pi = 16\arctan\frac{1}{5} - 4\arctan\frac{1}{239},$$

which leads to the series

$$\pi = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(\frac{16}{5^{2k+1}} - \frac{4}{239^{2k+1}}\right)$$

which converges much faster. Using the following variant on this theme, that

$$\pi = 48\arctan\frac{1}{49} + 128\frac{1}{57} - 20\frac{1}{239} + 48\frac{1}{110443}$$

was used by Yasumasa Kanada of Tokyo University in 2002 to compute π to 1,241,100,000,000 digits.

For a modern method there is the formula found in 1995 by Simon Plouffe:

$$\pi = \sum_{k=0}^{\infty}\frac{1}{16^k}\left(\frac{4}{8k+1} - \frac{2}{8k+4} - \frac{1}{8k+5} - \frac{1}{8k+6}\right)$$

[1] If you wish to prove this, probably the easiest way is to notice that $(5+i)^4(239-i) = 114244(1+i)$ and use the polar form of complex numbers to get the result.

which nice form the point of view of computing as powers of 16 are very easy to compute in hexadecimal. In particular using the first n terms of this series gives at least the first n-hexadecimal digits of π.