Mathematics 551 Take Home part of Test 1.

This is due at the beginning of class on Wednesday, February 12.

Problem 1. (20 points) We have shown that if $\mathcal{L}(p,\theta)$ is the line

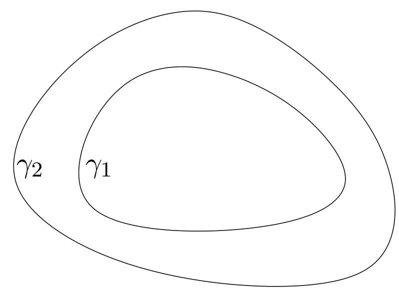
$$\mathcal{L}(p,\theta) = \{(x,y) : x\cos(\theta) + y\sin(\theta) = p\}$$

then for any C^1 curve γ that ***Crofton's Fromula***

$$L(\gamma) = \frac{1}{2}\int_0^{2\pi}\int_{-\infty}^{\infty} \#(\gamma \cap \mathcal{L}(p,\theta))\,dp\,d\theta$$

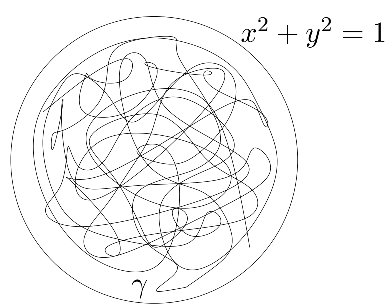
where $L(\gamma)$ is the length of γ.

(a) Let γ_1 and γ_2 be convex curves with γ_1 surrounded by γ_2 as in this figure:



Prove $L(\gamma_1) < L(\gamma_2)$.

(b) Let γ be a curve of length 1,000 that is contained inside of the unit circle $x^2 + y^2 = 1$:



Show there is a line that intersects γ at least 160 times.
Hint: $1000/(2\pi) = 159.1549\ldots$. □

Problem 2. (20 points) Let $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ be a unit speed curve and assume the curvature satisfies $\kappa(s) > 0$. Let $r > 0$ and $\mathbf{t}$ and $\mathbf{n}$ be the unit tangent and unit normal to $\mathbf{c}$. Then the ***parallel curve*** at distance r from $\mathbf{c}$ is the curve

$$\mathbf{c}_r = \mathbf{c}(s) - r\mathbf{n}(s).$$

(a) If the original curve $\mathbf{c}$ is a circle of radius R traversed so that the curvature is positive, draw picture and explain why $\mathbf{c}_r$ is the circle with radius $R+r$. (This explains why it is natural to use the negative in the definition of $\mathbf{c}_r$.)

(b) Show

$$\frac{d\mathbf{c}_r}{ds} = \mathbf{c}_r'(s) = (1 + r\kappa(s))\mathbf{t}(s)$$

and use this to explain why the unit tangent and normal to $\mathbf{c}_r$ are just $\mathbf{t}(s)$ and $\mathbf{n}(s)$. Draw a picture to illustrate this.

(c) Use Part (b) to show that length of $\mathbf{c}_r$ is

$$L(\mathbf{c}_r) = L(\mathbf{c}) + r\int_a^b \kappa(s)\, ds.$$

(d) Compute the curvature of $\mathbf{c}_r$. *Hint:* We know the unit tangent and normal to $\mathbf{c}_r$. Let σ be arclength along $\mathbf{c}_r$. Then

$$\frac{d\sigma}{ds} = \|\mathbf{c}_r'(s)\| = (1 + r\kappa(s)).$$

□

Things you should know for the in class part of the test.

(1) The basic definitions and formulas for things such as length, velocity, speed, and curvature.

(2) You definitely need to know the Frenet formulas.

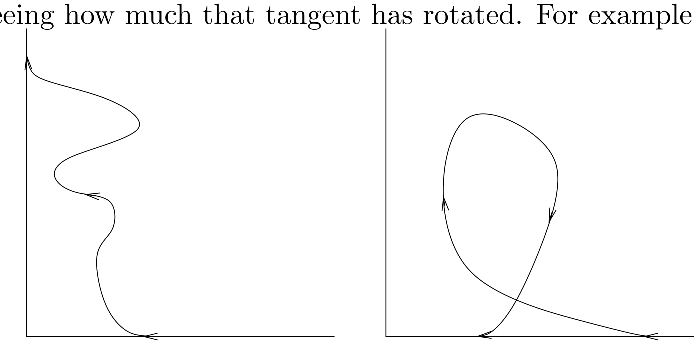
(3) Be able to use that curvature is

$$\kappa = \frac{d\theta}{ds}$$

which implies that for a unit speed curve

$$\int_a^b \kappa(s)\, ds = \int_a^b \frac{d\theta}{ds}\, ds = \theta(b) - \theta(a).$$

This allows one to be able to compute integrals of curvature by just seeing how much that tangent has rotated. For example



In the first curve above θ changes form π to $\pi/2$ and so for this curve $\int \kappa(s)\, ds = -\pi/2$. For the second curve the tangent goes through one resolution in the negative direction and so for this curve $\int \kappa(s)\, ds = -2\pi$.

(4) Know that statement of the isoperimetric inequality $4\pi A \leq L^2$.

(5) Know the statement of the maximum principle. A reasonable question would be to outline a proof of the four vertex therm using the maximum principle.