Mathematics 551 Homework, March 22, 2020

One of the topics we have not talked about is the area of surfaces. We start with the example of a parameterized plane. Let $\mathbf{P}_0 \in \mathbb{R}^3$ and let $\mathbf{a}$ and $\mathbf{b}$ be vectors in $\mathbb{R}^3$ with $\mathbf{a} \times \mathbf{b} \neq \mathbf{0}$. Deriving the basic formula for the area is based on the formula for the area of a parallelogram whose sides are the vector $\mathbf{v}$ and $\mathbf{w}$ is

$$\text{Area of parallelogram} = \|\mathbf{v} \times \mathbf{w}\|. \tag{1}$$

If you want to review this here is a video that should help:
`https://www.youtube.com/watch?v=W9dv3Vkf8mg`

Define $\mathbf{f}\colon \mathbb{R}^2 \to \mathbb{R}^3$ by

$$\mathbf{f}(u, v) = \mathbf{P}_0 + u\mathbf{a} + v\mathbf{b}.$$

Problem 1. Let $\mathbf{N} = \mathbf{a} \times \mathbf{b}$ (this vector is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$). Show that $\mathbf{f}$ satisfies

$$(\mathbf{f} - \mathbf{P}_0) \cdot \mathbf{N} = 0$$

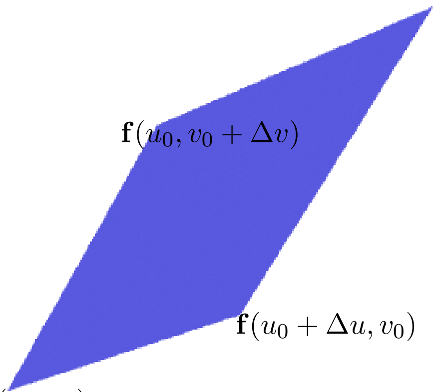
and therefore $\mathbf{f}$ parameterizes a plane. □

Now let us consider the rectangle

$$R = \{(u, v) : u_0 \leq u \leq u_0 + \Delta u,\ v_0 \leq v \leq v_0 + \Delta v\}$$

where Δu and Δv are positive numbers. The image of this rectangle under $\mathbf{f}$ is parallelogram defined by

$$\mathcal{P} = \{\mathbf{P}_0 + u\mathbf{a} + v\mathbf{b} : u_0 \leq u \leq u_0 + \Delta u,\ v_0 \leq v \leq v_0 + \Delta v\}$$



Problem 2. The sides of this parallelogram are the vectors

$$\mathbf{f}(u_0 + \Delta u, v_0) - \mathbf{f}(u_0, v_0) = \Delta u\, \mathbf{a}$$
$$\mathbf{f}(u_0, v_0 + \Delta v) - \mathbf{f}(u_0, v_0) = \Delta v\, \mathbf{b}$$

(a) Use Equation (1) to show the area of this parallelogram is

$$A = \|\mathbf{a} \times \mathbf{b}\|\, \Delta u\, \Delta v.$$

(b) Show that the partial derivatives of $\mathbf{f}$ are

$$\mathbf{f}_u = \mathbf{a} \qquad \mathbf{f}_v = \mathbf{b}$$

and therefore the area of the parallelogram can be written as

$$A = \|\mathbf{f}_u \times \mathbf{f}_v\|\, \Delta u\, \Delta v.$$

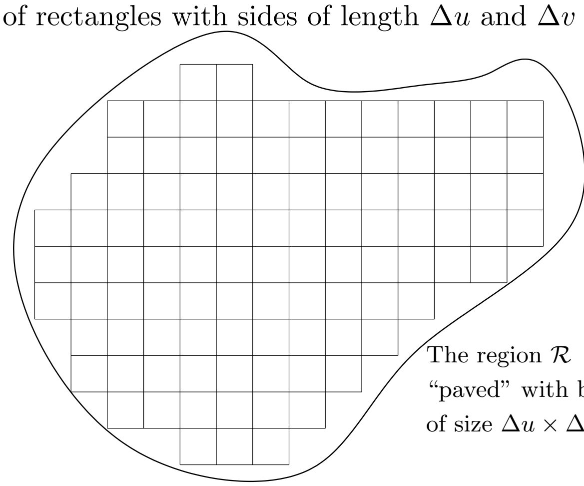
That is the area of the image of the parallelogram is obtained by multiplying the area of the rectangle by the number $\|\mathbf{a} \times \mathbf{b}\|$.

(c) Letting R explain why the area A can also be written as

$$A = \iint_R \|\mathbf{f}_u \times \mathbf{f}_v\|\, du\, dv.$$

Hint: As $\|f_u \times \mathbf{f}_v\|$ is constant this case, follows from basic the formula $\iint_D c\, du\, dv = c\, \mathrm{Area}(D)$ where D is any bounded domain and c is a constant (you can assume this formula). This is formula is given here as it is the form that generalizes to curved surfaces. $\square$

Now consider a bounded region $\mathcal{R}$ in the uv-plane. We want to compute the area of the image of $\mathcal{R}$ under the map $\mathbf{f}$. We have just seen how to do this when the region is a rectangle with its sides parallel to the axes. For a general region, $\mathcal{R}$, choose small positive real numbers Δu and Δv fill it up with a bunch of rectangles with sides of length Δu and Δv as shown:



The region $\mathcal{R}$ "paved" with bricks of size $\Delta u \times \Delta v$.

I like to think of this as having an irregular region in a yard that we want to make into a patio by paving with bricks. For bricks of size $\Delta u \times \Delta v$ we will not be able to full up the region, but by using smaller and smaller bricks we can get closer and closer to filling the region up.

To compute the are of the image, $\mathbf{f}[\mathcal{R}]$, of $\mathcal{R}$ under $\mathbf{f}$ let $R_1, R_2, \ldots, R_n$ be a paving of $\mathcal{R}$ by $\Delta u \times \Delta v$ rectangle as in the above figure. If Δu and Δv are small enough then these rectangle almost fill $\mathcal{R}$ and so also the images $\mathbf{f}[R_1], \mathbf{f}[R_2], \ldots, \mathbf{f}[R_n]$ almost fill the image $\mathbf{f}[\mathcal{R}]$. Therefore

$$\begin{aligned}
\operatorname{Area}(\mathbf{f}[\mathcal{R}]) &\approx \sum_{j=1}^{n} \operatorname{Area}(\mathbf{f}[R_j]) \\
&= \sum_{j=1}^{n} \iint_{R_j} \|\mathbf{f}_u \times \mathbf{f}_v\| \, du \, dv \\
&\approx \iint_{\mathcal{R}} \|\mathbf{f}_u \times \mathbf{f}_v\| \, du \, dv.
\end{aligned}$$

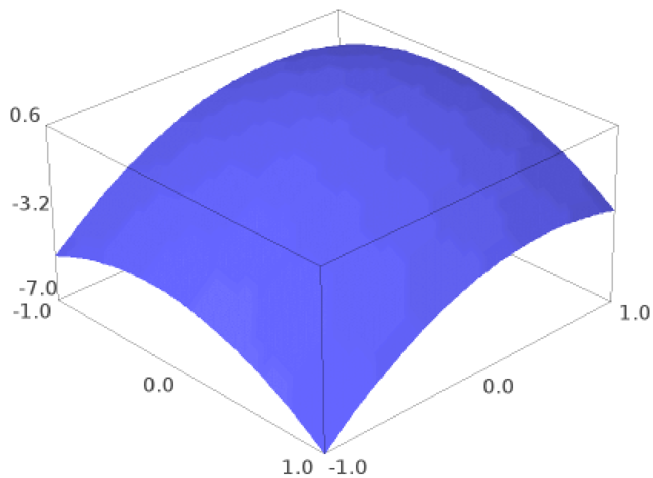
By taking the limit as both $\Delta u, \Delta v \to 0$ we arrive at

$$\operatorname{Area}(\mathbf{f}[\mathcal{R}]) = \iint_{\mathcal{R}} \|\mathbf{f}_u \times \mathbf{f}_v\| \, du \, dv.$$

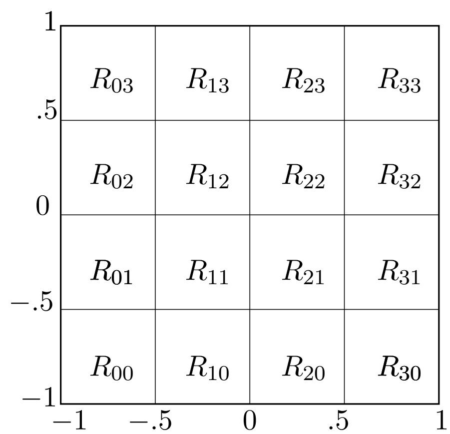
We now compute the area of a curved surface. To be concrete we look at the surface M parameterized by the map $\mathbf{x}\colon [-1,1] \times [-1,1] \to \mathbb{R}^3$ by

$$\mathbf{x}(u,v) = (u, v, -u + 2v - 2u^2 - 2v^2).$$

Here is a picture:



To get a first approximation of the area pave the rectangle $\mathcal{R} = [-1, 1] \times [-1, 1]$ with 16 bricks of size $\Delta u \times \Delta v = .5 \times .5$:



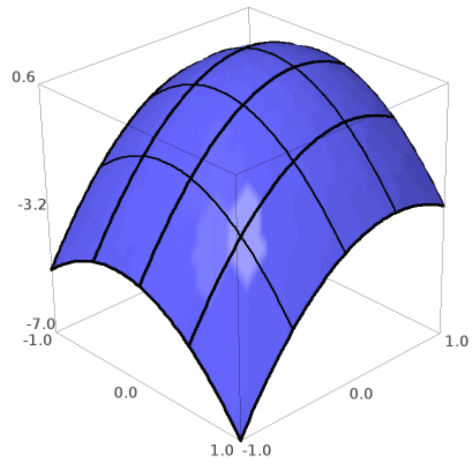
Let for $0/lej, k \leq 3$ let R_{jk} be the sub-rectangle (i.e. brick)

$$R_{jk} = [-1 + .5j, -1 + .5(j+1)] \times [-1 + .5k, -1 + .5(k+1)]$$

and let

$$P_{jk} = (-1 + .5j, -1 + .5k] = (a_j, b_k)$$

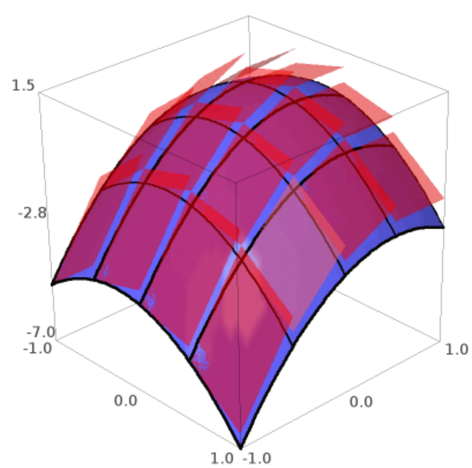
be the lower left corner of this rectangle (and this defines a_j and b_k). These rectangles are shown in the figure above. The images of these little rectangles on the surface M look like:



The tangent plane to M at the point $\mathbf{x}(P_{jk}) = \mathbf{x}(a_j, b_k)$ is parameterized by the linear map

$$\mathbf{f}_{jk}(u, v) = \mathbf{x}(a_j, b_k) + \mathbf{x}_u(a_j, b_k)(u - a_j) + \mathbf{x}_v(a_j, b_k)(v - b_k).$$

Then if the rectangles are small, then the image $\mathbf{f}_{jk}[R_{jk}]$ will be close to the image $\mathbf{x}[R_{jk}]$. Here is a picture of the image $\mathbf{x}[\mathcal{R}] = M$ together will all of the images $\mathbf{f}_{jk}[R_{jk}]$.



Hopefully this figure shows that the area of the surface M is very close to the area of the little red parallelograms that are tangent to it. Since the maps $\mathbf{f}_{jk}$ are linear know from our work above that

$$\operatorname{Area}(\mathbf{f}_{jk}[R_{jk}]) = \iint_{R_{jk}} \|\mathbf{x}_u(a_j, b_k) \times \mathbf{x}_v(a_j, b_k)\| \, du \, dv.$$

Adding these up gives

$$\operatorname{Area}(M) \approx \sum_{j,k} \iint_{R_{jk}} \|\mathbf{x}_u(a_j, b_k) \times \mathbf{x}_v(a_j, b_k)\| \, du \, dv.$$

But when $(u, v) \in R_{jk}$ the point (u, v) is close to (a_j, b_k) and therefore

$$\iint_{R_{jk}} \|\mathbf{x}_u(a_j, b_k) \times \mathbf{x}_v(a_j, b_k)\| \, du \, dv \approx \iint_{R_{jk}} \|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)\| \, du \, dv$$

So we end up with

$$\begin{aligned} \operatorname{Area}(M) &\approx \sum_{j,k} \iint_{R_{jk}} \|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)\| \, du \, dv \\ &= \iint_{\mathcal{R}} \|\mathbf{x}_u(u, v) \times \mathbf{x}_v(u, v)\| \, du \, dv. \end{aligned}$$

By making the paving rectangles smaller and smaller (that is taking a limit) we end up with

Theorem 1. *Let $\mathbf{x} \colon U \to \mathbb{R}^3$ be a C^2 regular map. Then the area of the image of $\mathbf{x}$ is*

$$\operatorname{Area}(\mathbf{x}[U]) = \iint_U \|\mathbf{x}(u, v) \times \mathbf{x}(u, v)\| \, du \, dv. \qquad \square$$

Now that we use this to find some areas.

Problem 3. Let $U \subseteq \mathbb{R}^2$ be a bounded open set and $f \colon U \to \mathbb{R}$ a C^2 function. Then the graph of f is parameterized by

$$\mathbf{x}(x, y) = (x, y, f(x,)).$$

Show that the area of this graph is given by

$$\text{Area} = \iint_U \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy. \qquad \square$$

In some of the following problems we will using the notation and formulas (which we have used before)

$$\begin{aligned} E_1(t) &= (\cos(t), \sin(t), 0) \\ E_2(t) &= (-\sin(t), \cos(t), 0) \\ E_3(t) &= (0, 0, 1) \\ E_1'(t) &= E_2(t) \\ E_2'(t) &= E_1(t) \\ E_3'(t) &= \mathbf{0}. \end{aligned}$$

Problem 4. You very likely looked at the area of surfaces of revolution in calculus. Here we look at this from the point of view of Theorem 1. . Let

$$\mathbf{c}(u) = (x(u), y(u))$$

with $x(u) > 0$ and $a < u < b$ be a C^2 curve in the plane. Define

$$\mathbf{f}(u, v) = x(u)E_1(v) + y(u)E_3 \qquad a < u < b \quad 0 \leq v \leq 2\pi$$

This is the result of revolving $\mathbf{c}$ around the y-axis in $\mathbb{R}^3$. Show that

$$\mathbf{f}_u \times \mathbf{f}_v = x(u)\left(-y'(u)E_1(v) + x'(u)E_3\right)$$

and therefore the area of the surface of revolution is

$$A = 2\pi \int_a^b x(u)\sqrt{x'(u)^2 + y'(u)^2}\, du$$

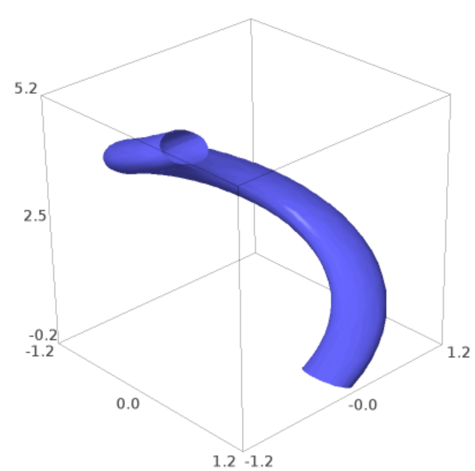
□

Problem 5. Here is more interesting example. Let $\mathbf{c}\colon [0, L]$ be a unit speed curve in $\mathbb{R}^3$ and let $r > 0$ be a positive number such that the curvature of $\mathbf{c}$ satisfies $\kappa(s) < 1/r$ where κ is the curvature of $\mathbf{c}$. (Since $\mathbf{c}$ is unit speed L is the length of the curve.) Then the ***tube of radius*** r ***around*** $\mathbf{c}$ is the surface parameterized by

$$\mathbf{f}(s, t) = \mathbf{c}(s) + r(\cos(t)\mathbf{n}(s) + \sin(t)\mathbf{b}(s))$$

where $0 \leq s \leq L$ and $0 \leq t \leq 2\pi$. Here is a picture of the tube of radius $r = 1/4$ about the helix

$$\mathbf{c}(s) = \left(\cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2}\right), \qquad 0 \leq s \leq 5.$$



Show

$$\|\mathbf{f}_s \times \mathbf{f}_t\| = |1 - r\cos(t)\kappa(s)|$$

and use that $\kappa < 1/r$ to show $1 - r\cos(t)\kappa(s)$ is positive and therefore

$$\|\mathbf{f}_s \times \mathbf{f}_t\| = 1 - r\cos(t)\kappa(s)$$

(i.e. the absolute values are not needed). Use this to show

$$A = 2\pi r L.$$

This is a little surprising, as the area only depends on the length of the curve, and not on its curvature and/or torsion. $\square$