Mathematics 551 Homework, April 3, 2020

Now let $L(t,x,y)$ be a smooth (this time smooth means that the first and second partial derivatives exist and are continuous) of (t,x,y). Let $u\colon [a,b]\to \mathbb{R}$ be a function such that for all other functions v on $[a,b]$ that
(1)

$$u(a)=v(a),\ u(b)=v(b) \implies \int_a^b L(t,u(t),\dot u(t))\,dt \le \int_a^b L(t,v(t),\dot v(t))\,dt.$$

That is u minimizes $\int_a^b L(t,u(t),\dot u(t))\,dt$ over all functions with the same boundary values as u. In this setting the function L is often called the ***Lagrangian***.

Our goal is to show that this implies any such minimizer will satisfy a certain differential equation. Toward this end let $g\colon [a,b]\to\mathbb{R}$ be any smooth function with

$$g(a)=g(b)=0.$$

Then for any real number ε the function

$$u_\varepsilon(t)=u(t)+\varepsilon g(t)$$

has $u_\varepsilon(a)=u(a)$ and $u_\varepsilon(b)=u(b)$. Therefore if we define a function of ε by

$$f(\varepsilon)=\int_a^b L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\,dt$$

and note that when $\varepsilon=0$ that $u_0=u$ we see that f has a minimum at $\varepsilon=0$. Therefore the derivative of f at $\varepsilon=0$ vanishes. That is

$$\begin{aligned} f'(0) &= \frac{d}{d\varepsilon}\int_a^b L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\,dt\bigg|_{\varepsilon=0} \\ &= \int_a^b \frac{d}{d\varepsilon}L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\bigg|_{\varepsilon=0}\,dt \\ &= 0 \end{aligned} \tag{2}$$

where we can move the derivative under the integral by a theorem of advanced calculus.

Problem 1. Use the chain rule to show that

$$\frac{d}{d\varepsilon}L(t,u_\varepsilon(t),\dot u_\varepsilon(t))\bigg|_{\varepsilon=0} = \frac{\partial L}{\partial x}(t,u(t),\dot u(t))g(t)+\frac{\partial L}{\partial y}(t,u(t),\dot u(t))\dot g(t).$$

(Here the notation is that $L=L(t,x,y)$ is a function of (t,x,y). Thus $\frac{\partial L}{\partial x}$ is the partial derivative with respect to the second variable and $\frac{\partial L}{\partial y}$ the derivative with respect to the third. It is also common to write these as $\frac{\partial L}{\partial u}$ and $\frac{\partial L}{\partial \dot u}$.)

Using this in the equation (2) we get

$$0 = \int_a^b \left(\frac{\partial L}{\partial x}(t, u(t), \dot{u}(t)) g(t) + \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) \dot{g}(t) \right) dt \tag{3}$$

$$= \int_a^b \frac{\partial L}{\partial x}(t, u(t), \dot{u}(t)) g(t) dt + \int_a^b \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) \dot{g}(t) dt$$

Problem 2. Use integration by parts and that $g(a) = g(b) = 0$ to show

$$\int_a^b \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) \dot{g}(t) dt = -\int_a^b \left(\frac{d}{dt} \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) \right) g(t)\, dt.$$

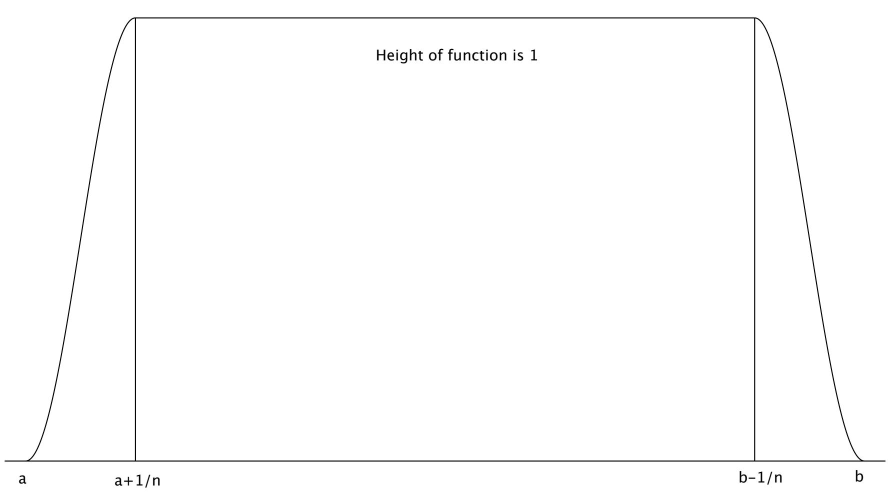
Problem 3. Combine the last problem with equation (3) to conclude

$$\int_a^b \left(\frac{d}{dt} \frac{\partial L}{\partial y}(t, u(t), \dot{u}(t)) - \frac{\partial L}{\partial x}(t, u(t), \dot{u}(t)) \right) g(t)\, dt = 0$$

for all smooth g on $[a, b]$ with $g(a) = g(b) = 0$.

We will assume the following.

Lemma 1. *For any interval $[a, b]$ and any positive integer with $1/n < (b - a)/2$ there is a smooth (in this setting this means the first derivative exists and is continuous) function, $\phi_{a,b,n}(t)$ that looks like*



That is $\phi_{a,b,n}(t)$ is zero at the endpoints $t = a$ and $t = b$ of the interval, has the value 1 for $a + 1/n \leq t \leq b - 1/n$ and $0 \leq \phi_{a,b,n}(t) \leq 1$. □

If you don't want to assume this you can check that

$$\phi_{a,b,n}(t) = \begin{cases} \sin^2(n(t-a)\pi/2), & a \leq t \leq a + 1/n; \\ 1, & a + 1/n \leq t \leq b - 1/n; \\ \sin^2(n(b-t)\pi/2), & b - 1/n \leq t \leq b \end{cases}$$

does the trick.

Proposition 2. *Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function with a continuous derivative on the interval $[a,b]$ such that for all functions, g, that are continuous functions with continuous derivatives on $[a,b]$ and with $g(a) = g(b) = 0$ that*

$$\int_a^b f(t)g(t)\,dt = 0.$$

Then $f(t) = 0$ for all $t \in [a,b]$.

Problem 4. Prove this. *Hint:* Let for any n the function $g_n(t) = \phi_{a,b,n}(t)f(t)$ is continuous with continuous derivative and $g_n(a) = g_n(b) = 0$. Therefore

$$\int_a^b f(t)g_n(t)\,dt = 0.$$

What happens when we take the limit as $n \to \infty$ in this?

We can now state the main result.

Theorem 3 (Euler-Lagrange Equation)**.** *Let u be a smooth function that solves the minimization problem* (1) *above. Then u satisfies the **Euler-Lagrange** equation*

$$\frac{d}{dt}\frac{\partial L}{\partial y}(t,u(t),\dot{u}(t)) - \frac{\partial L}{\partial x}(t,u(t),\dot{u}(t)) = 0.$$

Problem 5. Prove this. *Hint:* Let $f(t) = \frac{d}{dt}\frac{\partial L}{\partial y}(t,u(t),\dot{u}(t)) - \frac{\partial L}{\partial x}(t,u(t),\dot{u}(t))$. By Problem 3 we have

$$\int_a^b f(t)g(t)\,dt = 0$$

for all smooth $g(t)$ with $g(a) = g(b) = 0$. Now use Proposition 2. □

We will usually write the Euler-Lagrange equation as

$$\frac{d}{dt}L_{\dot{u}} - L_u = 0$$

rather than use the (slightly more correct) notation using L_y and L_x.

Theorem 4 (Conservation of Energy)**.** *If the Lagrangian, L, is independent of t, that is $L = L(u,\dot{u})$ has no explicit dependence on t, then for any solution of the Euler-Lagrange equation the quantity*

$$E = \dot{u}L_{\dot{u}} - L$$

*is constant. (For reason coming from physics this is called the **energy**.*

Proof. The are assuming that

$$\frac{d}{dt}L_{\dot{u}}(u,\dot{u}) - L_u(\dot{u},u) = 0.$$

To show E is constant, it is enough to show its derivative is zero.

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}\left(\dot{u}L_{\dot{u}} - L\right) \\ &= \ddot{u}L_{\dot{u}} + \dot{u}\left(\frac{d}{dt}L_{\dot{u}}\right) - \left(L_{\dot{u}}\ddot{u} + L_u\dot{u}\right) \\ &= \dot{u}\left(\frac{d}{dt}L_{\dot{u}} - L_u\right) \\ &= \dot{u}\cdot(0) \\ &= 0\end{aligned}$$

□

Problem 6. Let $L(t,x,y) = \sqrt{1+y^2}$ and let the interval be $[a,b] = [0,1]$. Then for a function $u(t)$ on these interval

$$\int_0^1 L(t,u(t),\dot{u}(t))\,dt = \int_0^1 \sqrt{1+\dot{u}(t)^2}\,dt$$

is just the length of the graph of u. Find the Euler-Lagrange equation for this integral and show that the solutions are straight lines. □