First collection of Final Problems in Math 551

Send me the solutions to these problems by 5:00pm on Monday, April 20. Use as the subject line Subject:

Math 551, Problems 1 and 2, ⟨your name⟩.

Your solution should be LaTeX output. I will be happy to answer questions related to these question on Wednesday and Friday. And if you have LaTeX questions you can e-mail me about them (use the word LaTeX in the subject line if you are doing this)

A topic in the classical theory that has received a good deal of attention is umbilic points. First a bit of review to fix notation. If M is a surface in $\mathbb{R}^3$ with unit normal $\mathbf{n}$ then the shape operator, S, is the linear map on tangent spaces defined defined on a tangent vector $\vec{u}$ by letting $S(\vec{u})$ be the negative of the directional derivative of $\mathbf{n}$ in the direction $\vec{u}$.

$$S(\vec{u}) = -D_{\vec{u}}\mathbf{n}.$$

This map is symmetric (also called self-adjoint). That is for tangent vectors $\vec{u}, \vec{v} \in T_pM$

$$\langle S\vec{u}, \vec{v}\rangle = \langle \vec{u}, S\vec{v}\rangle.$$

Then the eigenvalues, k_1 and k_2 of S are real and are the ***principle curvatures*** of M. At a point p where $k_1 = k_2$, or what is the same thing, S is a scalar multiple of the identity:

$$S = kI$$

where in this case $k = k_1 = k_2$ is the common value of the principle curvatures at p. You can find a nice summary of this, and a good review of the basics we have done with surface theory in the first 18 minutes of this video by Farid Tari. (He uses somewhat different notation than we have the main difference is that he writes the shape operator as $d\mathbf{n}$ rather than S.)

We have proven:

Theorem 1. *If $S = kI$ where k is a constant, then either M is part of a plane (when $k = 0$) or a standard round sphere (when $k \neq 0$).* □

This still leave the possibility that every point is an umbilic point, $S = kI$, but k is not constant. The following shows this can not happen.

Theorem 2. *Let $\mathbf{x}\colon U \to \mathbb{R}^3$ be parameterization of a smooth surface and assume that the shape operator satisfies $S = kI$ where $k = k(u, v)$ is a function on U. Then k is constant and therefore the surface is either part of a plane or part of a round sphere.*

Problem 1. Prove this along the following lines:

(a) The shape operator is defined by

$$S(\mathbf{x}_u) = -\frac{\partial \mathbf{n}}{\partial u}$$
$$S(\mathbf{x}_v) = -\frac{\partial \mathbf{n}}{\partial v}$$

Use and $S = kI$ to show

$$\frac{\partial \mathbf{n}}{\partial u} = -k\mathbf{x}_u \tag{1}$$
$$\frac{\partial \mathbf{n}}{\partial v} = -k\mathbf{x}_v \tag{2}$$

(b) Take $\frac{\partial}{\partial v}$ of equation (1) and $\frac{\partial}{\partial u}$ of equation (2) to get

$$\frac{\partial^2 \mathbf{n}}{\partial v \partial u} = -\frac{\partial k}{\partial v}\mathbf{x}_u - k\mathbf{x}_{uv}$$
$$\frac{\partial^2 \mathbf{n}}{\partial u \partial v} = -\frac{\partial k}{\partial u}\mathbf{x}_v - k\mathbf{x}_{vu}$$

(c) Now use that partial derivative commute so that,

$$\frac{\partial^2 \mathbf{n}}{\partial v \partial u} = \frac{\partial^2 \mathbf{n}}{\partial u \partial v} \qquad \text{and} \qquad \mathbf{x}_{uv} = \mathbf{x}_{vu},$$

to conclude

$$\frac{\partial k}{\partial v}\mathbf{x}_u = \frac{\partial k}{\partial u}\mathbf{x}_v.$$

(d) Use that $\mathbf{x}_u$ and $\mathbf{x}_v$ are linearly independent to conclude

$$\frac{\partial k}{\partial v} = \frac{\partial k}{\partial u} = 0$$

and use the to complete the proof. □

Problem 2. Let $\mathbf{x}\colon U \to \mathbb{R}^3$ parameterize a regular surface. Then the area of the image of $\mathbf{x}$ is

$$\text{Area}(\mathbf{x}) = \iint_U \|\mathbf{x}_u \times \mathbf{x}_v\|\, du\, dv.$$

Let $\mathbf{n}$ be the unit normal to this surface and r are real number. Define a new map $\mathbf{f}\colon U \to \mathbb{R}^3$ by

$$\mathbf{f}(u, v) = \mathbf{x}(u, v) + r\mathbf{n}(u, v).$$

This is the ***parallel surface at a distance*** r. We assume that

$$\mathbf{f}_u \times \mathbf{f}_v \neq \mathbf{0}.$$

(a) Show

$$\mathbf{f}_u = \mathbf{x}_u - S(X_u)$$
$$\mathbf{f}_v = \mathbf{x}_v - S(X_v)$$

where S is the shape operator of $\mathbf{x}$.

(b) Show the area of $\mathbf{f}$ is

$$\text{Area}(\mathbf{f}) = \text{Area}(\mathbf{x}) - 2r \iint_U H\, dA + r^2 \iint_U K\, dA$$

where $dA = \|\mathbf{x}_u \times \mathbf{x}_v\|\, du\, dv$ is the area measure on the original surface and H and K area are the mean and Gauss curvature of the original surface. $\square$