Mathematics 551 Homework, February 1, 2020

First a trick or two to that make some calculations easier. Let

$$\mathbf{e}_1(\theta) = (\cos(\theta), \sin(\theta))$$
$$\mathbf{e}_2(\theta) = (-\sin(\theta), \cos(\theta))$$

These vectors are both unit vectors (that is have $\|\mathbf{e}_j\| = 1$, and are orthogonal (i.e. $\mathbf{e}_1 \cdots \mathbf{e}_2 = 0$). Also $\mathbf{e}_1 \wedge \mathbf{e}_2 = 1$. Also useful is that their derivatives are

$$\mathbf{e}_1' = \mathbf{e}_2$$
$$\mathbf{e}_2' = -\mathbf{e}_1.$$

Often writing curves in terms of this basis simplifies calculations it avoids a good deal (but not all) algebra and having to simplify expressions involving trigonometric functions. To be a little more explicit about how this works if we write two vectors $\mathbf{v}$ and $\mathbf{w}$ in the basis $\mathbf{e}_1$ and $\mathbf{e}_2$ as

$$\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2$$
$$\mathbf{w} = w_1\mathbf{e}_1 + w_2\mathbf{e}_2$$

Then the following hold:

$$\|\mathbf{v}\|^2 = v_1^2 + v_2^2$$
$$\|\mathbf{w}\|^2 = w_1^2 + w_2^2$$
$$\mathbf{v} \cdot \mathbf{w} = v_1w_1 + v_2w_2$$
$$\mathbf{v} \wedge \mathbf{w} = \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} = v_1w_2 - v_2w_1.$$

Note if you wrote out $\mathbf{v}$ in the standard basis ($(1,0)$ and $(0,1)$) it is $\mathbf{v} = (v_1\cos(\theta) - v_2\sin(\theta), v_1\sin(\theta) + v_2\cos(\theta))$ and computing directly that $\|\mathbf{v}\|^2 = v_1^2 + v_2^2$ involves a fair amount of algebra and using that $\sin^2 + \cos^2 = 1$ at least twice. Likewise for the other formulas.

To look at an example of this consider a curve whose equation in polar coordinates is

$$r = \rho(\theta)$$

Using that the x and y of rectangular coordinates are related to the r and θ of polar coordinates by the x and y of rectangular coordinates are related by

$$x = r\cos(\theta) \tag{1}$$
$$y = r\sin(\theta) \tag{2}$$

we have that a parameterization of the curve in rectangular coordinates is

$$\mathbf{c}(\theta) = (\rho(\theta)\cos(\theta), \rho(\theta)\sin(\theta)).$$

This can be written as more compactly as

$$\mathbf{c} = \rho \mathbf{e}_1 \tag{3}$$

(to keep the notation shorter, and easier to read, we are suppressing θ, but keep in mind that $\rho = \rho(\theta)$ and $\mathbf{e}_1 = \mathbf{e}_1(\theta)$ depend on θ.)

Problem 1 (Arclength in polar coordinates). Verify the following: With $\mathbf{c}$ given by Equation (3) show that the velocity vector is

$$\mathbf{v} = \mathbf{c}' = \rho' \mathbf{e}_1 + \rho \mathbf{e}_1.$$

and therefore the speed is

$$v = \|\mathbf{c}'\| = \sqrt{\rho^2 + (\rho')^2}.$$

Therefore the part of the curve with $\alpha \le \theta \le \beta$ has length

$$L = \int_\alpha^\beta \sqrt{\rho(\theta)^2 + \rho'(\theta)^2}\, d\theta.$$

□

Problem 2. This problem is mostly a bit of review of calculus. Let $a > 0$ and consider the curve with polar equation

$$r = 2a\cos(\theta).$$

(a) Show that this is the circle with rectangular equation $(x-a)^2+y^2 = a^2$ and thus this circle has center $(a, 0)$ and radius a.
(b) Show that as θ moving from $-\pi/2$ to $\pi/2$ corresponds to moving around the circle once. (Proof by picture is fine for this.)
(c) Use these facts and the previous problem to show that the length of a circle of radius a is $2\pi a$. (I admit this is not the best way to see this, but it is a good test case for seeing we have the correct formula for the length.) □

Problem 3 (Curvature in polar coordinates). This continues Problem 1. Show that

$$\mathbf{c}'' = (\rho'' - \rho)\mathbf{e}_1 + 2\rho' \mathbf{e}_2$$

and therefore

$$\mathbf{c}' \wedge \mathbf{c}'' = \rho^2 + 2(\rho')^2 - \rho\rho''.$$

Recalling that curvature is given by

$$\kappa = \frac{\mathbf{c}' \wedge \mathbf{c}''}{v^3}$$

put the pieces together to show that in the case at hand

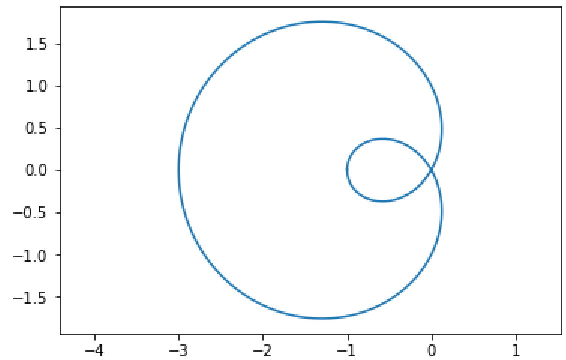
$$\kappa = \frac{\rho^2 + 2(\rho')^2 - \rho\rho''}{(\rho^2 + (\rho')^2)^{3/2}}.$$

□

Problem 4. The curve with polar equation

$$r = 1 - 2\cos(\theta)$$

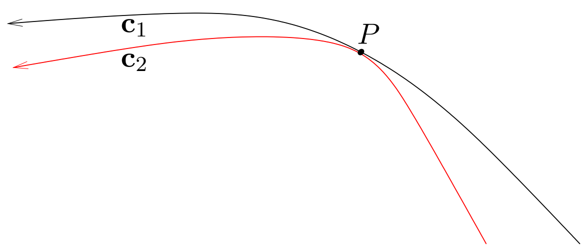
has the following graph:



Compute the curvature, $\kappa(\theta)$, of this curve and show that it has a maximum when $\theta = 0$, a minimum when $\theta = \pi$, and κ has no other local maximums or minimums. □

The following is the major result of the last couple of lectures.

Theorem 1 (The weak maximum principle)**.** *Let $\mathbf{c}_1$ and $\mathbf{c}_2$ be C^2 curves that go through the point P with the same unit tangent at P and with $\mathbf{c}_2$ to the left of $\mathbf{c}_1$ near P as in the follwing figure:*



Then the curvatures at P satisfy the inequality

$$\kappa_1(P) \leq \kappa_2(P).$$ □

Problem 5. We used Maximum principle in Osserman's proof of the Four Vertex Theorem. Review that proof and explain why the proof fails in the case of the curve in Problem 4 □

Problem 6. Let $0 < a < b$ and let

$$\mathbf{c}(t) = (a\cos(t), b\sin(t))$$

with $0 \leq t \leq 2\pi$ be a parameterization of the ellipse

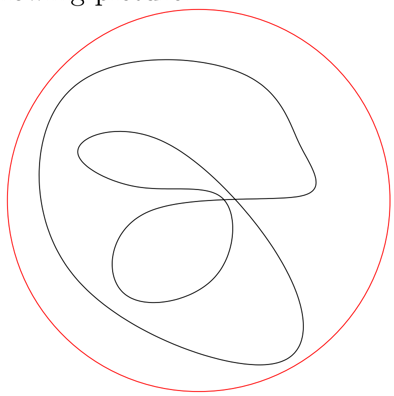
$$\frac{x^2}{a^2} + \frac{y^2}{b^2}.$$

Compute the curvature and show that this curve has has exactly four vertices. Where are they? □

Problem 7. Show that if a closed curve is contained in a circle of radius R that there is a point on the circle where the curvature satisfies

$$|\kappa| \geq \frac{1}{R}.$$

Hint: Consider the following picture:

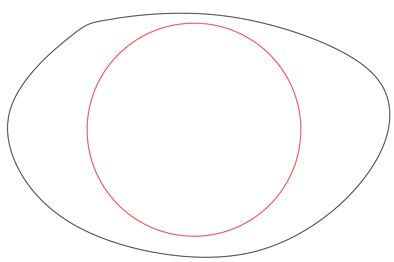


We can slide the outer circle (which has radius R) until it is tangent to the curve. You then have to worry a bit about which direction around the curve you are moving. We assume that the we are going counterclockwise around the circle (as this makes the curvature positive). You may have to reverse the direction you are moving around the curve so that that tangent to the curve and the tangent to the circle point in the same direction. This is why there is an absolute value in the inequality. □

Problem 8. Let **c** be a simple closed curve traversed in the positive (that is counterclockwise) direction and assume that **c** has positive curvature at all points. Assume that **c** surrounds a circle of radius R. Show that there is a point on **c** where the curvature satisfies

$$\kappa \leq \frac{1}{R}.$$

Hint: The picture



should give you a good idea of how to proceed. □