Mathematics 551 Homework, February 6, 2020

Let $c\colon [a,b] \to \mathbb{R}^2$ be a C^3 curve with curvature κ everywhere positive. Then the radius of curvature for the curve at $\mathbf{c}(s)$ is

$$\rho(s) = \frac{1}{\kappa(s)}.$$

The point

$$\mathbf{E}(s) = \mathbf{c}(s) + \rho(s)\mathbf{n}(s)$$

is the ***center of curvature*** of $\mathbf{c}$ at $\mathbf{c}(s)$ and the ***osculating circle*** at $\mathbf{c}(s)$ is the circle with center $\mathbf{E}(s)$ and radius $\rho(s)$. This is the circle that is tangent to $\mathbf{c}$ at $\mathbf{c}(s)$ and has the same curvature as the curve and therefore is the circle that "best fits" $\mathbf{c}$ at $\mathbf{c}(s)$. The curve $\mathbf{E}\colon [a,b] \to \mathbb{R}^2$ is the ***evolute*** of $\mathbf{c}$.

Problem 1. Use the Frenet formulas to show that

$$\mathbf{E}'(s) = \rho'(s)\mathbf{n}(s).$$

Then use this to show that the unit normal $\mathbf{n}_E(s)$ to $\mathbf{E}$ is

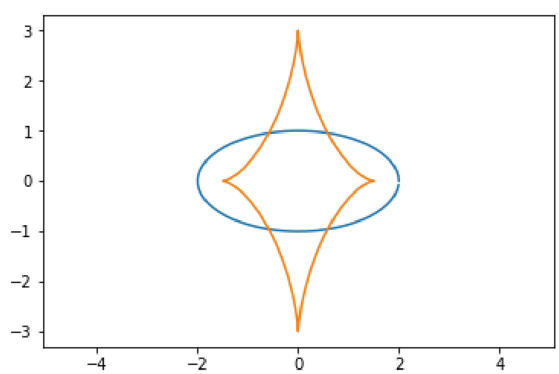
$$\mathbf{t}_E(s) = \begin{cases} \mathbf{n}(s), & \rho'(s) > 0; \\ -\mathbf{n}(s), & \rho'(s) < 0. \end{cases}$$

Use this to show that $\mathbf{n}_E$ flips direction by π radians at any point there ρ has a local maximum or minimum. That is at the vertices of $\mathbf{c}$. $\square$

Here is a picture of the ellipse

$$\frac{x^2}{2^2} + y^2 = 1$$

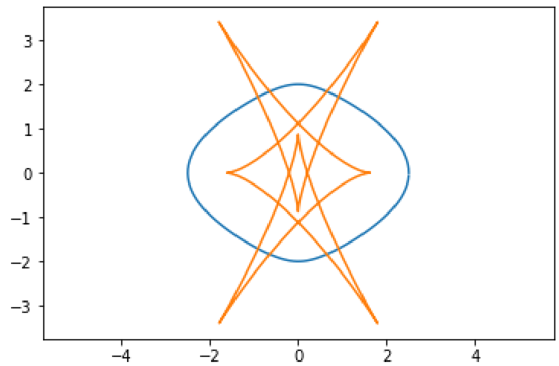
together with its evolute.



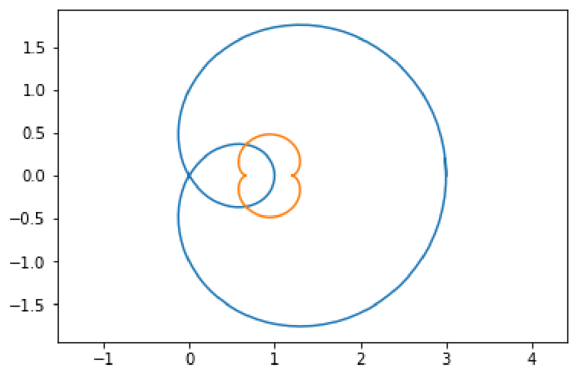
For a more exotic example, or at least one with more vertices, here is the graph of the curve

$$\mathbf{c}(t) = ((2 + .5\cos(2 * t))\cos(t), 2\sin(t)) \qquad 0 \le t \le 2\pi.$$

together with its evolute.



As one last example recall that curve with polar equation $r = 1 + 2\cos(\theta)$ was our example of a closed, but not simple, curve that only has two vertices. Thus its evolute should only have two cusps. Here is the picture showing this is the case:



Problem 2. Let $\mathbf{c}\colon [a, b] \to \mathbb{R}^2$ be a curve where $\kappa > 0$ and is monotone (that is either increasing or decreasing) on the interval. Show that the length of the evolute $\mathbf{E}$ is

$$\operatorname{Length}(\mathbf{E}) = |\rho(b) - \rho(a)|.$$

Hint: The arclength formula is

$$\operatorname{Length}(\mathbf{E}) = \int_a^b \|\mathbf{E}'(s)\| \, ds.$$

Now use that $\mathbf{E}'(s) = \rho'(s)\mathbf{n}$ and that since κ, and therefore also ρ, is monotone that ρ' is either always positive or always negative. $\square$

Problem 3. With the same hypothesis as the previous problem, show

$$\|\mathbf{E}(b) - \mathbf{E}(a)\| < |\rho(b) - \rho(a)|.$$

Hint: For a curve that is not a line segment, its length is greater than the distance between its endpoints. Or put more succinctly, the shortest path between two points is a straight line. $\square$

Problem 4. Let $\mathbf{P}_1$ and $\mathbf{P}_2$ be points in the plane, R_1, R_2 positive real numbers, and let $\mathcal{C}_j$ be the circle with center $\mathbf{P}_j$ and radius R_j. Show that if $\|\mathbf{P}_1 - \mathbf{P}_2\| < |R_1 - R_2|$ then one of the circles $\mathcal{C}_1$ or $\mathcal{C}_2$ is contained in the other one. *Hint:* Proof by picture is fine, and even preferred. $\square$

Theorem 1 (Tait-Kneser Theorem[1]). *Let $\mathbf{c}\colon [a,b] \to \mathbb{R}^3$ be a C^3 unit speed curve that has positive curvature. Also assume that κ is monotone. Then the osculating circles of the curve are nested. That if for any pair of them, one is contained inside the other.*

Problem 5. Prove this. *Hint:* Follow the outline of what we did in class. $\square$

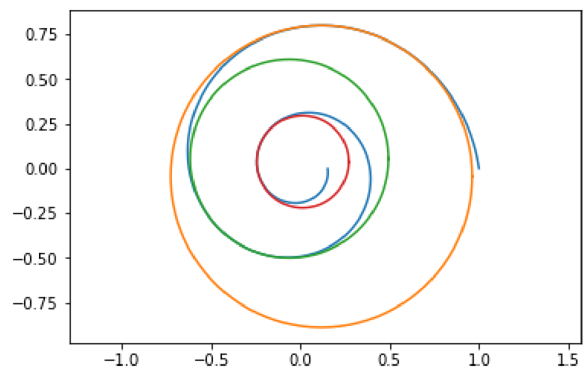
The following figure shows the curve $\mathbf{c}\colon [0,4\pi] \to \mathbb{R}^2$ given by

$$\mathbf{c}(t) = (e^{-.15t}\cos(t), e^{-.15}\sin(t))$$

(which has polar equation $r = e^{-.15\theta}$) which has curvature

$$\kappa(t) = \frac{20e^{\cdot 15t}}{\sqrt{409}}$$

which is increasing. Three of the osculating circles of the curve are shown.



The Tait-Kneser Theorem has some nice consequences. Note that a curve with positive curvature can cross itself many times.

Problem 6. Draw a curve with positive curvature that crosses itself four times. $\square$

Problem 7. Show that a curve $\mathbf{c}\colon [a,b] \to \mathbb{R}^2$ that has positive increasing curvature is embedded (the term ***embedded*** in this context just means that the curve does not cross itself). $\square$

Problem 8. We have show that for a simple closed curve of length L that encloses an area A and with inradius R_{in} and circumradius R_{out} that the

[1] This result was orginially proven by Peter Tait in a paper published in 1896. Adolf Kneser rediscovered it and published a proof in 1912.

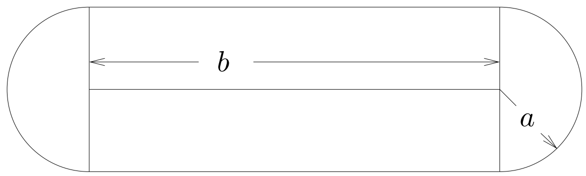
Bonnesen*[2] *inequalities

$$\frac{L - \sqrt{L^2 - 4\pi A}}{2\pi} \le R_{\text{in}} \le R_{\text{out}} \le \frac{L + \sqrt{L^2 - 4\pi A}}{2\pi}$$

and

$$\pi^2 (R_{\text{out}} - R_{\text{in}})^2 \le L^2 - 4\pi A$$

hold.



In the figure above we have $R_{\text{in}} = a$. Compute A and L for this shape and show

$$R_{\text{in}} = a = \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi}$$

and therefore the lower bound for R_{in} is best possible. $\square$

[2]These were first appeared in a paper by the Danish mathematician Tommy Bonnesen in 1921