

Mathematics 551 Homework, February 26, 2020

Problem 1. This problem is about rotating a quadratic to eliminate the x - y term. Let

$$q(x, y) = Ax^2 + 2Bxy + Cy^2.$$

The substitution

$$\begin{aligned}x &= \bar{x} \cos \theta + \bar{y} \sin \theta \\y &= -\bar{x} \sin \theta + \bar{y} \cos \theta\end{aligned}$$

corresponds to rotating the axis by an angle of θ and where \bar{x} and \bar{y} are new variables.

(a) Using this substitution show

$$q(x, y) = q(\bar{x} \cos \theta + \bar{y} \sin \theta, -\bar{x} \sin \theta + \bar{y} \cos \theta) = \bar{A}\bar{x}^2 + 2\bar{B}\bar{x}\bar{y} + \bar{C}\bar{y}^2$$

where

$$\begin{aligned}\bar{A} &= A \cos^2 \theta - 2B \cos \theta \sin \theta + C \sin^2 \theta \\ \bar{B} &= A \cos \theta \sin \theta + B(\cos^2 \theta - \sin^2 \theta) - C \cos \theta \sin \theta \\ \bar{C} &= A \sin^2 \theta + 2B \cos \theta \sin \theta + C \cos^2 \theta\end{aligned}$$

(b) Use the trigonometric identities $\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$ and $\sin(2\theta) = 2 \cos(\theta) \sin(\theta)$ to rewrite the equation for \bar{B} as

$$\bar{B} = \frac{1}{2}(A - C) \sin(2\theta) + B \cos(2\theta).$$

(c) Show that if $B \neq 0$ and θ is chosen so that

$$\cot(2\theta) = \frac{C - A}{2B}$$

then $\bar{B} = 0$. Thus if we set $\bar{q}(\bar{x}, \bar{y}) = q(\bar{x} \cos \theta + \bar{y} \sin \theta, -\bar{x} \sin \theta + \bar{y} \cos \theta)$ we have

$$\bar{q}(\bar{x}, \bar{y}) = \bar{A}\bar{x}^2 + \bar{C}\bar{y}^2$$

That is this rotation eliminates the cross term $\bar{x}\bar{y}$. □

In Shifrin's book read the first part of Section 2.1, page 35 to the top of page 39. We will use the notation

$$\begin{aligned}E_1(\theta) &= (\cos \theta, \sin \theta, 0) \\ E_2(\theta) &= (-\sin \theta, \cos \theta, 0) \\ E_3(\theta) &= (0, 0, 1).\end{aligned}$$

Then for all θ these are an orthonormal basis of \mathbb{R}^3 , that is

$$\begin{aligned} E_i \cdot E_i &= 1 & \text{for } 1 \leq i \leq 3 \\ E_i \cdot E_j &= 0 & \text{for } 1 \leq i < j \leq 3 \\ E_1 \times E_2 &= E_3 \\ E_2 \times E_3 &= E_1 \\ E_3 \times E_1 &= E_2. \end{aligned}$$

Problem 2. In this notation Shifrin's Example 1 (c) on page 36 (the torus) has the parameterization

$$\mathbf{x}(u, v) = (a + b \cos u)E_1(v) + b \sin u E_3 \quad 0 \leq u \leq 2\pi, \quad 0 \leq v \leq 2\pi.$$

- (a) Draw a picture of the image of \mathbf{x} (which is just Figure 1.3 in Shifrin) and draw the curve

$$\boldsymbol{\gamma}(t) = \mathbf{x}(t, t) \quad 0 \leq t \leq 2\pi.$$

- (b) Find the first fundamental form of \mathbf{x} .
(c) Set up the integral for the length of $\boldsymbol{\gamma}$. (Don't try to evaluate this integral.) \square

Problem 3. Let $\mathbf{c}(s) = (x(s), y(s))$ with $0 \leq s \leq L$ be a unit speed curve in the plane. Define $\mathbf{x}: [0, L] \times (-\infty, \infty) \rightarrow \mathbb{R}^3$ by

$$\mathbf{x}(u, v) = (x(u), y(u), v).$$

Find the first fundamental form of \mathbf{x} . \square

Problem 4. For $u > 0$ and $0 \leq v \leq 2\pi$ and m a positive constant define

$$\mathbf{f}(u, v) = uE_1(v) + muE_3.$$

Draw a picture of the image of \mathbf{f} and find its first fundamental form. \square

Problem 5 (Optional). With the notation of Problem 1 show for any θ that

$$\overline{A}\overline{C} - \overline{B}^2 = AC - B^2 \quad \text{and} \quad \overline{A} + \overline{C} = A + C \quad \square$$